MATH 271, Quiz 4

Due November 13^{TH} at the end of class

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page, but no other online resources (including calculators or WolframAlpha) for this quiz. **Do not discuss any problem any other person.** All of your solutions should be easily identifiable and supporting work must be shown. Ambiguous or illegible answers will not be counted as correct.

THERE ARE 4 PROBLEMS AND 2 BONUS PROBLEMS.

Problem 1. Consider the matrix

$$[A] = \begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}.$$

(a) **(3 pts.)** Show that for any arbitrary vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^2$ that

$$([A]\vec{\boldsymbol{u}}) \cdot \vec{\boldsymbol{v}} = \vec{\boldsymbol{u}} \cdot ([A]\vec{\boldsymbol{v}}).$$

(b) **(2 pts.)** Is $[A]$ hermitian? Explain.

Problem 2. For the following decide whether the statement is true or false. For full credit, provide an adequate explanation for your answer.

(a) **(2 pts.)** Every matrix is diagonalizable.
(b) **(2 pts.)** Every $n \times n$ matrix has n complex eigenvalues.
(c) **(2 pts.)** The trace satisfies $\mathrm{tr}([A][B][C]) = \mathrm{tr}([A])\mathrm{tr}([B])\mathrm{tr}([C])$.
(d) **(2 pts.)** Similar matrices have the same eigenvalues.

Problem 3. Consider the matrix

$$[A] = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) **(2 pts.)** Argue why the eigenvalues of $[A]$ must be real.
(b) **(2 pts.)** Find the eigenvalues of $[A]$.
(c) **(2 pts.)** Find the eigenvectors of $[A]$.
(d) **(2 pts.)** Show that the eigenvectors of $[A]$ are orthogonal.
(e) **(2 pts.)** Explain how you can use the eigenvectors of $[A]$ to transform $[A]$ into a similar diagonal matrix.

Problem 4. (3 pts.) Let A be a linear transformation and let $\vec{\boldsymbol{e}}_1, \ldots, \vec{\boldsymbol{e}}_k$ be eigenvectors all with corresponding eigenvalue λ. Show that any vector in the span of $\{\vec{\boldsymbol{e}}_1, \ldots, \vec{\boldsymbol{e}}_k\}$ is also an eigenvector with eigenvalue λ.

Problem 5. (Bonus 3 pts.) Consider the following set $\{1, -1, \mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}, \mathbf{k}, -\mathbf{k}\}$ with the relationships

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

where 1 and -1 behave as expected. Show that this set with multiplication is a group (called the *quaternion group*).

Problem 6. (Bonus 3 pts.) Consider the group of $n \times n$ invertible matrices denoted by $\mathrm{GL}(n)$. Explain why $\mathrm{GL}(n)$ is a group when the product operation is given by matrix multiplication but it is **NOT** a group if we instead chose the operation to be addition.