MATH 271, Quiz 2
Due March 19th at the end of class

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page, but no other online resources (including calculators or WolframAlpha) for this quiz. **Do not discuss any problem any other person.** All of your solutions should be easily identifiable and supporting work must be shown. Ambiguous or illegible answers will not be counted as correct.

THERE ARE 7 TOTAL PROBLEMS.

Problem 1. For the following, say whether the statement is true or false. For full credit, justify your answer with an explanation.

(a) **(2 pts.)** A partial differential equation must always have initial conditions.

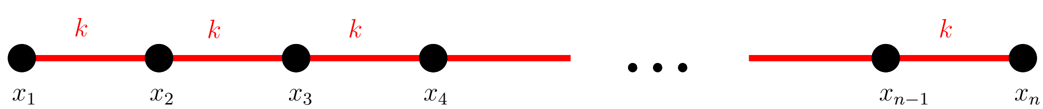
(b) **(2 pts.)** The Dirichlet boundary conditions for the heat equation specify a specific temperature at each point along the boundary of the domain.

(c) **(2 pts.)** The 1-dimensional wave equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0$$

with $c > 0$ will reach an equilibrium position as we let $t \to \infty$ regardless of the initial conditions $u(x,0)$ and $\frac{\partial}{\partial t}u(x,0)$.

Problem 2. Let us take a discrete set of points on a line like so



A discrete version of the 1-dimensional Poisson equation

$$-k\frac{d^2}{dx^2}u(x) = f(x),$$

can be written for each of the interior points as

$$-k\frac{u_{j+1} - 2u_j + u_{j-1}}{\delta x^2} = f(x_j).$$

Here, δx is the distance between each particle, $k \neq 0$ is the strength of a spring connecting each particle, u_j describes the height of the particle above the x-axis, and $f(x_j)$ is the force acting on particle j.

(a) **(2 pts.)** True or false. The force on particle j does not affect particle $j+2$. Explain.

(b) **(2 pts.)** Neumann boundary conditions prescribe values for

$$\frac{\partial}{\partial x}u(0) \qquad \text{and} \qquad \frac{\partial}{\partial x}u(1).$$

Explain why in this approximation we would instead prescribe a value for

$$\frac{u_1 - u_2}{\delta x} \qquad \text{and} \qquad \frac{u_n - u_{n-1}}{\delta x}.$$

Hint: think about what happens as you consider the continuum limit (i.e., $n \to \infty$).

(c) **(2 pts.)** In general, what are the equations for the boundary particles 1 and n?

Problem 3. (3 pts.) Consider the Helmholtz equation

$$\Delta f = -k^2 f,$$

where Δ is the Laplacian. Suppose that f_1 is a solution with $k = k_1$ and f_2 is a solution with $k = k_2$ and suppose further that $k_1 \neq k_2$. Show that

$$f = f_1 + f_2$$

is *not* a solution to the Helmholtz equation.

Problem 4. (BONUS 2 pts.) Argue that the previous problem would yield a solution if $k_1 = k_2$.

Problem 5. Consider the 1-dimensional heat equation.

(a) **(2 pts.)** Let the domain be $\Omega = [0,1]$ with the boundary conditions

$$\frac{\partial}{\partial x}u(0,t) = 0 \qquad \text{and} \frac{\partial}{\partial x}u(1,t) = 0.$$

Show that $u_n(x,t) = A_n e^{n^2\pi^2 t}\cos(n\pi x)$ satisfies the boundary conditions for any integer n.

(b) **(2 pts.)** Using your solution in (a), find the particular solution that satisfies the initial condition $u(x,0) = 13\cos(18\pi x)$.

Problem 6. (3 pts.) Find the solution to the wave equation on $\mathbb{R}$ with $c = 1$ using d'Alembert's formula with the initial conditions

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial t}u(x,0) = 1.$$

Problem 7. (BONUS 3 pts.) Let $u(x,t)$ be a solution to the heat equation with time dependent heat sources given by $f(x,t)$. That is, $u(x,t)$ solves the equation

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t) + f(x,t).$$

Show that equilibrium can only be achieved if $\lim_{t\to\infty} f(x,t) = g(x)$.