MATH 272, HOMEWORK 9
DUE MAY 3RD

Problem 1. Let $\Psi(x)$ be a complex function with domain $[0, L]$. Show that multiplication by a global phase $e^{i\theta}$ does not affect the norm of $\Psi(x)$ under the Hermitian (integral) inner product. In more generality, this shows that you cannot fully determine a quantum state – there will always be an undetermined phase. *For simplicity, use the inner product for the particle in the box.*

Problem 2. Consider the real function $f(x) = 1$ on the domain $[0, L]$.

(a) What is the norm of f, $\|f\|$?

(b) Normalize $f(x)$.

(c) Find a nonzero normalized polynomial of degree ≤ 1 that is orthogonal to $f(x)$.

Problem 3. A wavefunction $\Psi(x)$ for a particle in the 1-dimensional box $[0, L]$ could be written as a superposition of normalized states

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

That is,

$$\Psi(x) = \sum_{n=1}^{\infty} a_n \psi_n(x),$$

for some choice of the coefficients a_n.

(a) Let $a_n = \frac{\sqrt{6}}{n\pi}$. Show that $\Psi(x)$ is normalized. *Hint: first, use orthogonality of the states $\psi_n(x)$ to your advantage. Then you will need to know what an infinite series evaluates to. Use a tool like WolframAlpha to evaluate this series.*

(b) Note that we can approximate $\Psi(x)$ by taking a finite sum approximation up to some chosen N by

$$\Psi(x) \approx \sum_{n=1}^{N} a_n \psi_n(x).$$

Plot the approximation of $\Psi(x)$ for $N = 1, 5, 50, 100$. *Hint: you can modify my Desmos examples.*

Problem 4. When making a measurement of the position of the particle, we will use the *position operator* x. This is the same as the variable x in the original problem statement, but it is also an operator!

(a) Show that the position operator x is Hermitian.

(b) We can compute the expected position of a particle with wavefunction $\Psi(x)$ by computing

$$\mathbb{E}[x] = \langle \Psi, x\Psi \rangle.$$

Let $\Psi(x) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x)$, compute $\mathbb{E}[x]$. This value $\mathbb{E}[x]$ tells you where we expect to find the particle on average.

(c) In fact, any real valued function $V(x)$ of the position operator x is also Hermitian. Make a quick argument on why this must be true.

Problem 5. Another related operator is the *momentum operator* $p = -i\hbar\frac{d}{dx}$. Using integration by parts, show that this operator is Hermitian.

Problem 6. We can always take products, sums, and scalar multiples of operators to build new operators. For example, in classical physics, we have the kinetic energy

$$T = \frac{1}{2}m\vec{v} \cdot \vec{v},$$

where $\vec{v}$ is the velocity. In 1-dimension, this reduces to the familiar $\frac{1}{2}mv^2$. However, we can also rewrite this 1-dimensional equation using the momentum $p = mv$ which gives us the kinetic energy

$$T = \frac{p^2}{2m}.$$

Hence, we can define the quantum *kinetic energy operator* using the above equation and the definition of p from the previous problem.

(a) Show that $T = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}$.

(b) Make a quick argument that shows that T is Hermitian.

(c) Again, letting $\Psi(x) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x)$, compute $\mathbb{E}[T]$. The expected value $\mathbb{E}[\hat{T}]$ tells us what the observed energy will be on average. Yet, any time we measure a system we will find that energy must be one of the energy eigenvalues. Thus, for this wave function, this expected value should be the average between E_1 and E_2 which means that half the time we will measure the energy to be E_1 and half the time it will be E_2.

Problem 7. If we are given a potential (energy) $V(x)$ and the kinetic energy T, we can take their sum and form the total energy $T + V(x)$ which we call the *Hamiltonian*. Thus, in the quantum realm, we create the Hamiltonian operator $\hat{H}$ by

$$H = T + V(x).$$

(a) Show that the Hamiltonian operator is Hermitian. *Hint: you have already done the necessary work for this. You just need to combine it and show a few steps here.*

(b) The spectrum of the Hamiltonian tells us the possible energy eigenvalues of a quantum system. Thus, we can compute the spectrum (in this case) by solving the eigenvalue equation

$$H\Psi(x) = E\Psi(x).$$

Explain why the spectrum of H is discrete for the particle in the box problem. *Hint: We have done this exact problem before. Feel free to use that!*

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