MATH 272, HOMEWORK 4

DUE FEBRUARY 26TH

Problem 1. Plot each of the following vector fields. Describe what each field represents in the relevant coordinate system. Do we see any points in space where there are issues with these vector fields?

(a) $\hat{\boldsymbol{\rho}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}$.

(b) $\hat{\boldsymbol{\theta}} = \frac{-y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}$.

(c) $\hat{\boldsymbol{\phi}} = \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}}$.

(d) $\hat{\boldsymbol{r}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}}$.

Problem 2. Let us see some of the usefulness of cylindrical coordinates.

(a) Using the fact that $\hat{\boldsymbol{\theta}} = \frac{-y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}$, convert the magnetic field

$$\vec{\boldsymbol{B}} = -\frac{y}{2}\hat{\boldsymbol{x}} + \frac{x}{2}\hat{\boldsymbol{y}},$$

into cylindrical coordinates (i.e., only a function of ρ, θ, z, and $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$).

(b) Parameterize a curve $\vec{\boldsymbol{\gamma}}(t)$ that traces out the unit circle in the xy-plane in cylindrical coordinates.

(c) Compute the tangent vector $\dot{\vec{\boldsymbol{\gamma}}}(t)$ in cylindrical coordinates?

(d) Compute the following integral using cylindrical coordinates

$$\int_{\vec{\boldsymbol{\gamma}}} \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\gamma}}.$$

Problem 3. Convert the following integrals to integrals in cylindrical coordinates. Also, describe the region in which you are integrating over. Do not evaluate the integrals.

(a) $\displaystyle\int_{-1}^{1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xyz\,dxdydz.$

(b) $\displaystyle\int_{0}^{1}\int_{-z}^{z}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} x^2+y^2+z^2 dxdydz.$

Problem 4. Note that the Laplacian Δ in cylindrical coordinates is given by

$$\Delta f(\rho,\theta,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

Compute the Laplacian of

$$f(\rho,\theta,z) = \sqrt{\rho^2+z^2}\,z\cos(\theta).$$

Problem 5. In spherical coordinates (either implicitly or explicitly), parameterize the following objects.

(a) A solid sphere with radius 3.

(b) The surface of an infinite cone with a vertex angle of $\pi/4$.

(c) A latitudinal curve on the unit sphere at the latitude of 30° above the equator.

(d) A solid unit sphere with a cylinder of radius $1/2$ removed from the core.

Problem 6. Let us see some of the benefit of using spherical coordinates.

(a) Using the fact that

$$\hat{\boldsymbol{r}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}},$$

convert the vector field

$$\vec{\boldsymbol{E}}(x,y,z) = \begin{pmatrix} \frac{x}{(x^2+y^2+z^2)^{3/2}} \\ \frac{y}{(x^2+y^2+z^2)^{3/2}} \\ \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{pmatrix},$$

into spherical coordinates (i.e., only a function of r, θ, ϕ, and $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$).

(b) Parameterize the surface of a sphere of radius R (which we'll call Σ) as well as the outward normal vector $\hat{\boldsymbol{n}}$ and in spherical coordinates.

(c) Compute the following integral using spherical coordinates that we have found:

$$\iint_\Sigma \vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}}\,d\Sigma,$$

where $d\Sigma$ will be the area form in spherical coordinates.

Problem 7. Note that the Laplacian Δ in spherical coordinates is given by

$$\Delta f(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right).$$

Compute the Laplacian of

$$f(r,\theta,\phi) = r^2\cos(\theta)\cos(\phi).$$