MATH 272, HOMEWORK 9
DUE APRIL 20TH

Problem 1. Consider the 2-dimensional source free isotropic heat equation given by

$$\left(-k\Delta + \frac{\partial}{\partial t}\right) u(x,y,t) = 0,$$

with the domain Ω as the unit square in the xy-plane. Take as well the Dirichlet boundary conditions $u(x,y,t) = 0$ for x and y on the boundary of Ω.

(a) Show that $u_{mn}(x,y,t) = \sin(m\pi x)\sin(n\pi y)e^{-k(n^2+m^2)\pi^2 t}$ is a solution to the PDE and Dirichlet boundary conditions for any non-negative integers m and n.

(b) Show that a linear combination of solutions u_{mn} and u_{pq} is also a solution.

(c) For $m = n = 1$ and $k = 1$, plot the solution for the values $t = 0$, $t = 0.01$, $t = 0.1$ and $t = 1$. Explain what is physically happening as time moves forward.

(d) Explain what varying the value for the conductivity k does to the solution. Feel free to use plots to support your hypothesis.

(e) Explain the mathematical reason why increasing m and n causes the solution to converge to zero more quickly.

(f) Explain the physical reason why increasing m and n causes the solution to converge to zero more quickly. Plots may help support your reasoning.

Problem 2. Consider the 1-dimensional homogeneous Laplace equation given by

$$\frac{\partial^2}{\partial x^2} u_E(x) = 0,$$

with the domain Ω as the unit interval on the x-axis. Take the Dirichlet boundary conditions $u_E(0) = T_0$ and $u_E(L) = T_L$. Think of these values as the ambient temperature at the endpoints of the rod. These temperatures are constant since the ambient environment is so large.

(a) Find the particular solution to this Laplace equation.

(b) Suppose that $v(x,t)$ is a solution to the 1-dimensional source free isotropic heat equation with zero Dirichlet boundary values. Show that

$$u(x,t) = v(x,t) + u_E(x),$$

is a solution to the 1-dimensional source free isotropic heat equation with Dirichlet boundary values $u(0,t) = T_0$ and $u(L,t) = T_L$.

(c) From Problem 1, we know that $\lim_{t\to\infty} v(x,t) = 0$. Hence, show that the long time limit of a solution to the source free heat equation yields a solution to the Laplace equation.

(d) Argue why the equilibrium temperature profile of a rod can be found without solving the heat equation.

Problem 3. Using intuition from the previous problem, explain how one could solve the heat equation with a nonzero source term that only depends on x. In other words, how could one try to solve

$$\left(-k\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) u(x,t) = f(x),$$

Problem 4. Consider the 1-dimensional wave equation given by

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0,$$

with the domain Ω as the unit interval on the x-axis. We shall fix the string at each endpoint which requires $u(0,t) = 0$ and $u(1,t) = 0$ for all t. Take the initial condition as well to be a plucked string so that $u(x,0) = \sin(\pi x)$ and $\frac{\partial}{\partial t}u(x,0) = 0$.

(a) Use the separation of variables ansatz $u(x,t) = X(x)T(t)$ to get a new separation constant. This will give two ODEs: one will be in terms of $X(x)$ and the other will be in terms of $T(t)$.

(b) Use the boundary conditions and solve the ODE that is in terms of $X(x)$ which will simultaneously find the allowed values for the separation constant.

(c) Using these allowed values for the separation constant, find the solution for $T(t)$.

(d) Find the particular solution for $u(x,t)$ by matching the initial condition.

(e) Plot your solution for $x \in [0,1]$ and $t \in [0,\infty)$ (i.e., just plot up to a large value of t). In this case, compare your plots for $c = 1/2$ and $c = 1$.