

MATH 272, HOMEWORK 8
DUE APRIL 6TH

Problem 1. Plot each of the following vector fields.

(a) $\hat{\rho} = \frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\hat{y}$.

(b) $\hat{\theta} = \frac{-y}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y}$.

(c) \hat{z} .

Problem 2. Plot each of the following vector fields.

(a) $\hat{r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{y} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{z}$.

(b) $\hat{\theta} = \frac{-y}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y}$.

(c) $\hat{\phi} = \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{y} + \frac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\hat{z}$.

Problem 3. Consider the following vector field

$$\vec{E} = \frac{x}{(x^2 + y^2 + z^2)^{3/2}}\hat{x} + \frac{y}{(x^2 + y^2 + z^2)^{3/2}}\hat{y} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\hat{z},$$

which you can think of as the electric field of a positive point charge. We argued that this field \vec{E} is conservative in a previous homework problem. Specifically, $\vec{E} = \vec{\nabla}\phi$, for some scalar field ϕ . This follows from Faraday's law for static charges.

(a) Compute the integral

$$T = \int_{\vec{\gamma}} \vec{E} \cdot d\vec{\gamma} \quad \text{where} \quad \vec{\gamma}(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix},$$

and $a \leq t \leq b$ with a and b both greater than 0. Note that this integral T describes the gain in kinetic energy of a charged particle that moved along the path $\vec{\gamma}$.

(b) Equivalently, since \vec{E} is conservative, we have

$$T = \int_{\vec{\gamma}} \vec{E} \cdot d\vec{\gamma} = \phi(\vec{\gamma}(b)) - \phi(\vec{\gamma}(a)).$$

Show that this is true for the given vector field and potential. This shows that the choice of path does not matter; only the endpoints $\vec{\gamma}(a)$ and $\vec{\gamma}(b)$ matter.

(c) Argue why the integral around any closed curve must be zero.

Problem 4. Let us see some of the benefit of using spherical coordinates.

(a) Using the fact that

$$\hat{\mathbf{r}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{x}} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{y}} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{z}},$$

convert the vector field $\vec{\mathbf{E}}$ into spherical coordinates (i.e., only a function of r , θ , ϕ , and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$).

(b) Parameterize the surface of a sphere of radius R (which we'll call Σ) as well as the outward normal vector $\hat{\mathbf{n}}$ and in spherical coordinates.

(c) Compute the following integral using spherical coordinates that we have found:

$$\iint_{\Sigma} \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} d\Sigma,$$

where $d\Sigma$ will be the area form in spherical coordinates.

Problem 5. Note that the Laplacian Δ in cylindrical coordinates is given by

$$\Delta f(\rho, \theta, z) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

Compute the Laplacian of

$$f(\rho, \theta, z) = \sqrt{\rho^2 + z^2} z \cos(\theta).$$

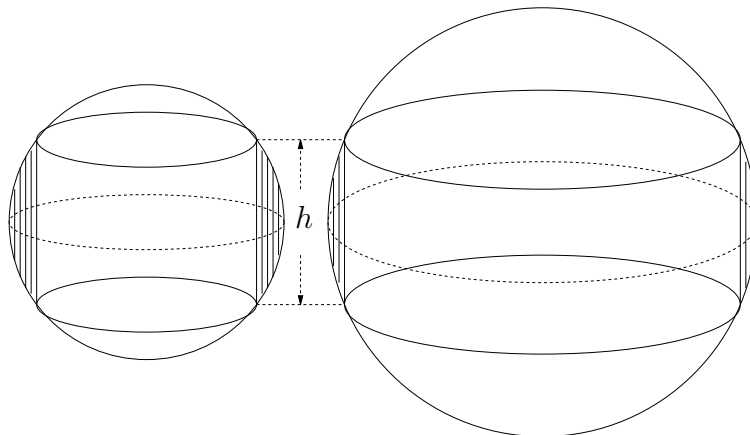
Problem 6. Note that the Laplacian Δ in spherical coordinates is given by

$$\Delta f(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial f}{\partial \phi} \right).$$

Compute the Laplacian of

$$f(r, \theta, \phi) = r^2 \cos(\theta) \cos(\phi).$$

Problem 7. (BONUS) The following problem is a somewhat pop-culture math paradox known as the *napkin ring problem* (see Vsauce for more). Consider the following problem. We want to compute the volume inside a ball of radius R after drilling out an inscribed cylinder of height h . See the following picture.



The question is, does the left over volume (of the napkin ring) depend on the radius R of the sphere. You have your choice of working in spherical or cylindrical coordinates. Use whichever helps you most.