MATH 272, Homework 7

Due March 31st

Problem 1. Plot each of the following vector fields.

(a) $\hat{\boldsymbol{\rho}} = \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}$.

(b) $\hat{\boldsymbol{\theta}} = \frac{-y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}$.

(c) $\hat{\boldsymbol{z}}$.

Problem 2. Consider the following vector field

$$\vec{\boldsymbol{B}} = -\frac{y}{2}\hat{\boldsymbol{x}} + \frac{x}{2}\hat{\boldsymbol{y}}.$$

Here, $\vec{\boldsymbol{B}}$ denotes the magnetic field. It may be helpful to plot the fields in this problem.

(a) Show that $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \vec{\boldsymbol{J}}$ (Ampére's law) where

$$\vec{\boldsymbol{J}} = \hat{\boldsymbol{z}}.$$

This vector field $\vec{\boldsymbol{J}}$ represents the electric current (moving charges) in space. One could argue that the current creates the magnetic field via Ampére's law.

(b) Magnetic fields induce a force $\vec{\boldsymbol{F}}$ on charged particles by the Lorentz force

$$\vec{\boldsymbol{F}} = \dot{\vec{\gamma}} \times \vec{\boldsymbol{B}} = \ddot{\vec{\gamma}}$$

Where $\dot{\vec{\gamma}}$ is the velocity of the particle (where we have chosen a mass $m = 1$ and charge $q = 1$). Let us do the following.

- Assume that $\dot{\vec{\gamma}} = \hat{\boldsymbol{x}}$, what is the force on the particle?
- Repeat the previous step for $\dot{\vec{\gamma}} = \hat{\boldsymbol{y}}$ and $\dot{\vec{\gamma}} = \hat{\boldsymbol{z}}$.
- Compare and contrast the forces you found.

(c) Can you argue why applying a magnetic field to a molecule may cause it to heat up? Can you compare this idea with your home microwave? (Please don't put anything dangerous in your microwave due to this discovery!)

Problem 3. Let us see some of the usefulness of cylindrical coordinates.

(a) Using the fact that $\hat{\boldsymbol{\theta}} = \frac{-y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}$, convert the magnetic field from Problem 2 into Cylindrical coordinates (i.e., only a function of ρ, θ, z, and $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$).

(b) Parameterize a curve $\vec{\gamma}(t)$ that traces out the unit circle in the xy-plane in cylindrical coordinates.

(c) What is the tangent vector $\dot{\vec{\gamma}}(t)$ in terms of $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$?

(d) Compute the following integral using cylindrical coordinates that we have found:

$$\int_{\vec{\gamma}} \vec{\boldsymbol{B}} \cdot d\vec{\gamma}$$

(e) Using cylindrical coordinates, compare your result with

$$\iint_{\Sigma} \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}\right) \cdot \hat{\boldsymbol{n}} d\Sigma,$$

where Σ is the unit disk.

Problem 4. Convert the following integrals to integrals in cylindrical coordinates. Also, describe the region in which you are integrating over. Do not evaluate the integrals.

(a) $\displaystyle \int_{-1}^{1} \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xyz dx dy dz.$

(b) $\displaystyle \int_{0}^{1} \int_{-z}^{z} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 + y^2 + z^2 dx dy dz.$

Problem 5. In spherical coordinates (either implicitly or explicitly), parameterize the following objects.

(a) A solid sphere with radius 3.

(b) The surface of an infinite cone with a vertex angle of $\pi/4$.

(c) A latitudinal curve on the unit sphere at the latitude of 30° above the equator.

(d) A solid unit sphere with a cylinder of radius $1/2$ removed from the core.