MATH 272, Homework 5

Due March 9th

Problem 1. A rough model of a molecular crystal can be described in the following way. Take the scalar function

$$u(x, y) = \cos^2(x) + \cos^2(y).$$

This function $u(x, y)$ describes the *potential energy* for electrons in the crystal. Electrons are attracted to the areas with the smallest potential energy and move away from areas of high potential energy.

(a) Plot this function and include a printout. Notice what this looks like. You can imagine that each of the low points (well) is where a nucleus is located in the crystal.

(b) Plot the level curves where $u(x, y) = 0$, $u(x, y) = \frac{1}{4}$, $u(x, y) = \frac{1}{2}$, and $u(x, y) = 1$ for the range of values $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$ and $-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$.

Picking the constant for the level curve tells you the *kinetic energy* of the electron you are looking at. It turns out that electrons (roughly) will orbit along these level curves. Notice, some level curves bleed into the different troughs of neighboring molecules which means that electrons of sufficient energy happily flow through the crystal. However, electrons like to behave a bit differently thanks to their quantum nature!

(c) Find the gradient of this function $\vec{\nabla} u(x, y)$.

(d) At what point(s) is the gradient zero? *Hint: Use your graph of the level curves to help.*

Problem 2. Let us visualize vector fields using GeoGebra (specifically `https://www.geogebra.org/m/u3xregNW`). Plot the following vector fields and print them out.

(a) (Constant wind from the northwest) $\vec{\boldsymbol{V}}(x, y) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

(b) (Two wind fronts meeting) $\vec{\boldsymbol{U}}(x, y, z) = \begin{pmatrix} y \\ x \\ 0 \end{pmatrix}$.

(c) (Source) $\vec{\boldsymbol{E}}(x, y, z) = \begin{pmatrix} \frac{x}{(x^2+y^2+z^2)^{3/2}} \\ \frac{y}{(x^2+y^2+z^2)^{3/2}} \\ \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{pmatrix}$.

(d) (Vortex) $\vec{\boldsymbol{S}}(x, y, z) = \begin{pmatrix} \frac{-y}{x^2+y^2+z^2} \\ \frac{x}{x^2+y^2+z^2} \\ 0 \end{pmatrix}$.

Problem 3. Compute the divergence and curl of the Source and Vortex fields from the previous problem. What can we say about the divergence and curl of these fields at the origin?

Problem 4. Consider the function

$$f(x, y) = \sin\left(\frac{2\pi x}{5}\right)\sin\left(\frac{2\pi y}{5}\right).$$

comes up when you want to find out how a square shaped drum head will vibrate when hit.

(a) Plot this function on the region Ω given by $0 \leq x \leq 5$ and $0 \leq y \leq 5$.

(b) What is the value the function $f(x, y)$ on the boundary of the given region Ω (i.e, when $x = 0$, $x = 5$, $y = 0$, and $y = 5$)?

(c) Show that $f(x, y)$ is an eigenfunction of the Laplacian $\Delta = \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}}$. What is the eigenvalue?

Problem 5. Consider the following vector field

$$\vec{\boldsymbol{E}}(x, y, z) = \begin{pmatrix} \frac{x}{(x^2+y^2+z^2)^{3/2}} \\ \frac{y}{(x^2+y^2+z^2)^{3/2}} \\ \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{pmatrix},$$

which models the electric field of an proton (in units of of charge $q = 1$) placed at the origin.

(a) Show that $\vec{\boldsymbol{E}}(x, y, z) = -\vec{\boldsymbol{\nabla}} V(x, y, z)$ where $V(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. We refer to $V(x, y, z)$ as the electrostatic potential or voltage.

(b) Let Ω be a box with side lengths two centered at the origin. Compute the total flux of $\vec{\boldsymbol{E}}$ through the surface of the box Σ. That is,

$$\int_\Gamma \vec{\boldsymbol{E}}(x, y, z) \cdot \hat{\boldsymbol{n}} d\Gamma.$$

(c) Does the total flux depend on the size or shape of the box?

(d) Using the provided argument, one can compute

$$\int_\Omega \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}(x, y, z) d\Omega.$$

- Compute $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}$ and note that this is zero everywhere except at $(x, y, z) = (0, 0, 0)$.
- Note that the two integrals in this problem are equal. This is known as the *divergence theorem* and it is a special case of a more general theorem called *Stokes' theorem* which generalizes the fundamental theorem of calculus. Hence, you can now argue why

  $$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = \delta(x, y, z),$$

  where $\delta(x, y, z)$ is the 3-dimensional Dirac delta.