MATH 272, Homework 3, *Solutions*

Due February 17$^{\text{th}}$

Problem 1. Compute the Fourier series for the following functions on the interval $[0, L]$. Then plot your result (for $N = 1, 50, 100, 500$) compared to the original function. What do you notice if you plot the Fourier series outside the range of $[0, L]$?

(a) $f(x) = \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{-4\pi x}{L}\right)$.

(b) $\sin\left(\frac{3\pi x}{L}\right)$.

(c) $\delta(x - L/2)$.

Solution 1.

(a) Here, we can use orthogonality to greatly reduce the amount of work we must do. Notice as well that our function is essentially written as a Fourier series. We have

$$\begin{aligned} a_n &= \left\langle f, \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right)\right\rangle \\ &= \left\langle \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{-4\pi x}{L}\right), \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right)\right\rangle \\ &= \underbrace{\left\langle \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right), \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right)\right\rangle}_{=0 \text{ unless n=1}} + \underbrace{\left\langle \sin\left(\frac{-4\pi x}{L}\right), \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right)\right\rangle}_{=0 \text{ always}}. \end{aligned}$$

Hence, we have that the only nonzero a_n term is a_1 and we have

$$\begin{aligned} a_1 &= \left\langle \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right), \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right)\right\rangle \\ &= \frac{1}{2\sqrt{2}}\left\langle \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right), \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right)\right\rangle \\ &= \frac{1}{2\sqrt{2}}. \end{aligned}$$

An analogous argument shows that $f(x)$ is orthogonal to 1 which shows that $a_0 = 0$ as well. Similarly, another analogous argument shows that the only nonzero b_n term is b_2 since $\sin(-x) = -\sin(x)$. So we have that

$$b_2 = \frac{-1}{\sqrt{2}}.$$

Hence, we have found all of our coefficients for the Fourier series.

(b) First, let's compute

$$\begin{aligned} a_0 &= \left\langle \sin\left(\frac{3\pi x}{L}\right), 1 \right\rangle \\ &= \frac{1}{L}\int_0^L \sin\left(\frac{3\pi x}{L}\right) dx \\ &= \frac{2}{3\pi}. \end{aligned}$$

Next, we can compute

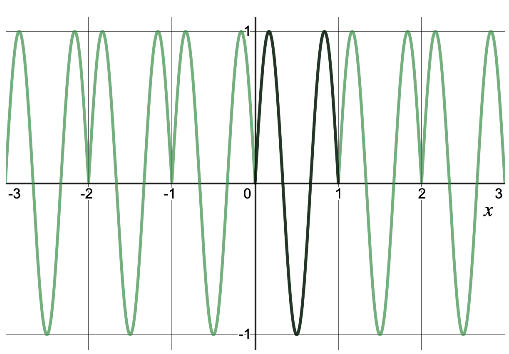
$$\begin{aligned} a_n &= \left\langle \sin\left(\frac{3\pi x}{L}\right), \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right) \right\rangle \\ &= \frac{1}{L}\int_0^L \sin\left(\frac{3\pi x}{L}\right) \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right) dx \\ &= \frac{6\sqrt{2}}{9\pi - 4\pi n^2}. \end{aligned}$$

Finally, we compute

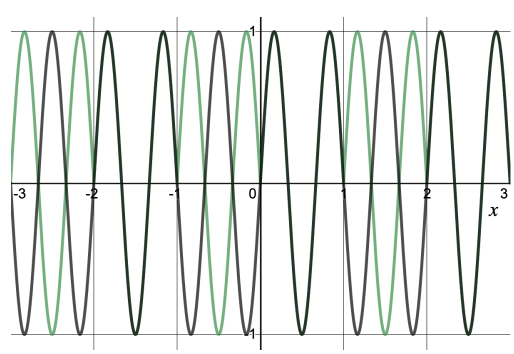
$$\begin{aligned} b_n &= \left\langle \sin\left(\frac{3\pi x}{L}\right), \sqrt{2}\sin\left(\frac{2n\pi x}{L}\right) \right\rangle \\ &= 0 \qquad\qquad \text{by orthogonality.} \end{aligned}$$

Thus, our Fourier series is given by

$$\boxed{\sin\left(\frac{3\pi x}{L}\right) = \frac{2}{3\pi} + \sum_{n=1}^{\infty} \frac{12}{9\pi - 4\pi n^2} \cos\left(\frac{2n\pi x}{L}\right).}$$



(a) Approximating $\sin\left(\frac{3\pi x}{L}\right)$ in black.



(b) Approximating $\sin\left(\frac{3\pi x}{L}\right)$ in black.

(c) First, we compute

$$a_0 = \langle \delta(x - L/2), 1 \rangle = \frac{1}{L}\int_0^L \delta(x - L/2) dx = \frac{1}{L}.$$

Then, we have

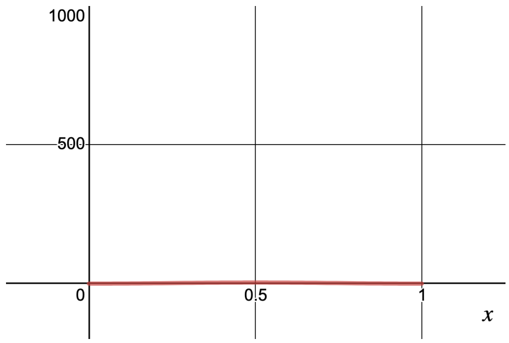
$$
\begin{aligned}
a_n &= \left\langle \delta(x - L/2), \sqrt{2}\cos\left(\frac{2n\pi x}{L}\right) \right\rangle \\
&= \frac{1}{L}\int_0^L \delta(x - L/2)\sqrt{2}\cos\left(\frac{2n\pi x}{L}\right) dx \\
&= \sqrt{2}L\cos(n\pi) \\
&= (-1)^n \frac{\sqrt{2}}{L}.
\end{aligned}
$$

Finally, we compute

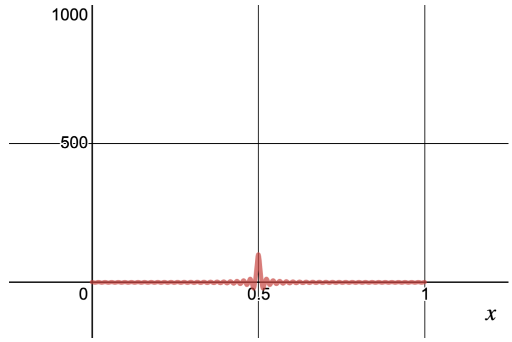
$$
\begin{aligned}
b_n &= \left\langle \delta(x - L/2), \sqrt{2}\sin\left(\frac{2n\pi x}{L}\right) \right\rangle \\
&= \frac{1}{L}\int_0^L \delta(x - L/2)\sqrt{2}\sin\left(\frac{2n\pi x}{L}\right) dx \\
&= \frac{\sqrt{2}}{L}\sin(n\pi) \\
&= 0.
\end{aligned}
$$

Hence, our Fourier series is given by

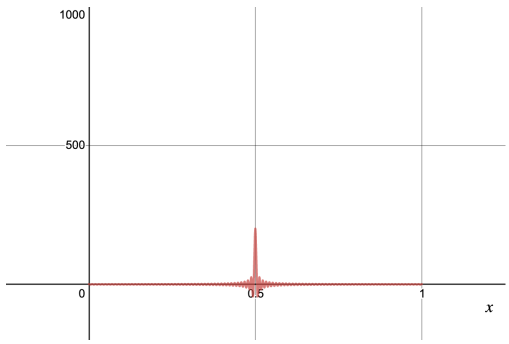
$$
\boxed{\delta(x - L/2)) = \frac{1}{L} + \sum_{n=1}^{\infty}(-1)^n \frac{2}{L}\cos\left(\frac{2n\pi x}{L}\right).}
$$



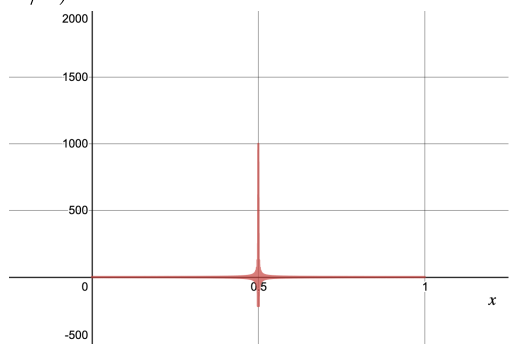
(a) $N = 1$ approximation to $\delta(x - L/2)$.



(b) $N = 50$ approximation to $\delta(x - L/2)$.



(c) $N = 100$ approximation to $\delta(x - L/2)$.



(d) $N = 500$ approximation to $\delta(x - L/2)$.

Problem 2. Consider a function $f(x)$ that describes the height of a rubber string with rest length L. We can attach the ends of the string at $x = 0$ and $x = L$ by requiring that $f(0) = f(L) = 0$. Then, one can subject the string to an external force $g(x)$ and find the profile of the string by solving

$$-\frac{d^2}{dx^2} f(x) = g(x).$$

(a) Let $g(x) = \delta(x - L/2)$ and let $f(x)$ be given by some Fourier series. Using the equation above, solve for the coefficients of the Fourier series for $f(x)$.

(b) Plot the Fourier series for $f(x)$ for $N = 1, 5, 50$.

This is an extremely important to solve. The fact that we can determine a solution $f(x)$ where the external force is the Dirac delta function means that we have the ability to determine a the deformation of a string from a point force.

Solution 2.

(a) Here we let $f(x)$ be written as an arbitrary Fourier series by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right).$$

Then we can put

$$-\frac{d^2}{dx^2} f(x) = \frac{4n^2\pi^2}{L^2}\left(\sum_{n=1}^{\infty} a_n \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right)\right).$$

Setting the left hand side of the ODE equal to the right, we have

$$-\frac{d^2}{dx^2} f(x) = g(x)$$

$$\frac{4n^2\pi^2}{L^2}\left(\sum_{n=1}^{\infty} a_n \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right)\right) = \frac{1}{L} + \sum_{n=1}^{\infty} (-1)^n \frac{2}{L} \cos\left(\frac{2n\pi x}{L}\right).$$

Hence, we just need to determine the coefficients a_0, a_n, and b_n for our function $f(x)$. It's clear to see that the b_n terms for $f(x)$ must be zero. Solving for the a_n terms, we find

$$\frac{4n^2\pi^2}{L^2} a_n = (-1)^n \frac{2}{\sqrt{2}L},$$

which yields

$$a_n = (-1)^n \frac{L}{2\sqrt{2}n^2\pi^2}.$$

At this point, we have that

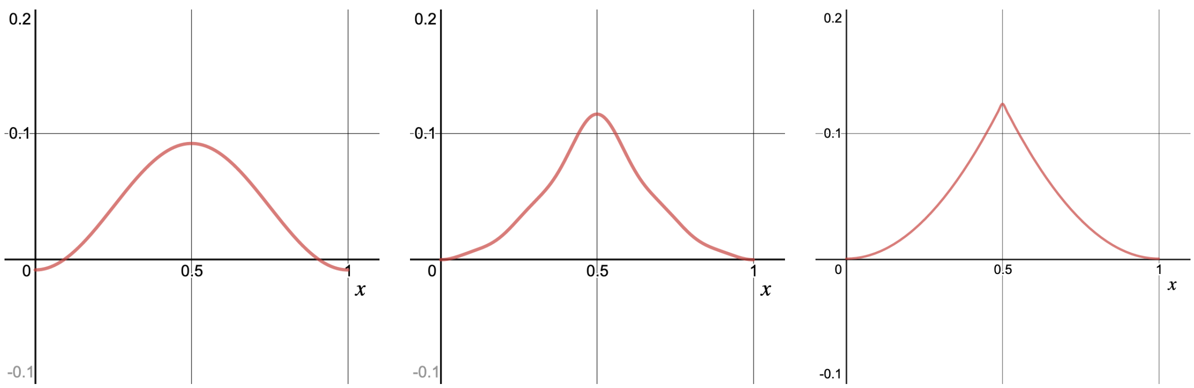
$$f(x) = a_0 + \sum_{n=1}^{\infty} (-1)^n \frac{L}{2n^2\pi^2} \cos\left(\frac{2n\pi x}{L}\right).$$

We can determine a_0 by inforcing our boundary conditions. Namely, we take

$$
\begin{aligned}
0 = f(0) &= a_0 + \sum_{n=1}^{\infty} (-1)^n \frac{L}{2n^2\pi^2} \\
&= a_0 - \frac{L}{24},
\end{aligned}
$$

where I used WolframAlpha to evaluate the infinite series above. This means that $a_0 = \frac{L}{24}$. Thus, we arrive at

$$
\boxed{f(x) = \frac{L}{24} + \sum_{n=1}^{\infty} (-1)^n \frac{L}{2n^2\pi^2} \cos\left(\frac{2n\pi x}{L}\right).}
$$



(a) $N = 1$ approximation to the differential equation..

(b) $N = 5$ approximation to the differential equation..

(c) $N = 50$ approximation to the differential equation..

Problem 3. Compute the following Fourier transforms (using a table or WolframAlpha if need be).

(a) $\sin(3\pi x)$.

(b) $e^{\frac{-x^2}{2}}$.

(c) $\delta(x)$.

Solution 3.

(a) To compute the Fourier transform, one could use a table. However, we can take

$$\mathcal{F}\left[\sin(3\pi x)\right] = \int_{-\infty}^{\infty} \sin(3\pi x) e^{-i2\pi kx} dx.$$

The techniques used to compute this integral by hand are found in complex analysis. Specifically, one uses the fact that $\sin(3\pi x)$ is an analytic function on $\mathbb{C}$ as well as the method of contour integration (seen here: `https://en.wikipedia.org/wiki/Contour_integration`). This is not a topic for our class (though we will learn how to integrate along curves).

Anyways, computing this Fourier transform yields

$$\boxed{\mathcal{F}\left[\sin(3\pi x)\right] = \frac{\delta\left(k - \frac{3}{2}\right) - \delta\left(k + \frac{3}{2}\right)}{2i}.}$$

Here, I used a table to compute this (as WolframAlpha claims the integral does not converge).

(b) Again, we want to compute

$$\mathcal{F}\left[e^{\frac{-x^2}{2}}\right] = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} e^{-i2\pi kx} dx.$$

I was able to compute this using WolframAlpha by entering

```
integrate[e^(-x^2/2)e^{-i2*pi*k*x},{x,-infty,infty}]
```

This yields

$$\boxed{\mathcal{F}\left[e^{\frac{-x^2}{2}}\right] = \sqrt{2\pi} e^{-2\pi^2 k^2}.}$$

One thing to note here is that the Fourier transform of a Gaussian is a Gaussian!

(c) Since, in some sense, the Dirac delta is a Gaussian function (with a variance of zero), we expect to get another type of Gaussian out as well. Let's see what happens. We take

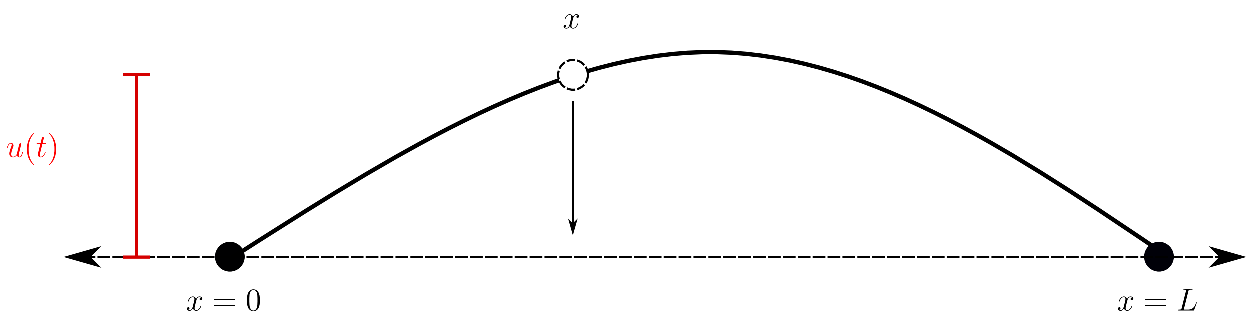
$$\boxed{\mathcal{F}\left[\delta(x)\right] = \int_{-\infty}^{\infty} \delta(x) e^{-i2\pi kx} dx = 1.}$$

The constant 1 function is much like a Gaussian with an infinite variance! This is maybe a "handwavy" way to look at this.

Problem 4. A common application for the Fourier transform is to solve differential equations whose domain is time $t \in [0, \infty)$. We can model how a point x on a rubber string oscillates over time consider the differential equation

$$u''(t) + v^2 u(t) = 0,$$

with initial conditions $u(0) = L$ and $u'(0) = 0$. Here $u(t)$ is the displacement of the string at position x with the initial conditions describing the string being pulled tight at time $t = 0$.



To solve this equation, we could use use methods we learned previously, or apply the Fourier transform to the whole equation by

$$\mathcal{F}\left[u''(t) + v^2 u(t)\right] = \mathcal{F}\left[0\right].$$

(a) Compute the Fourier transform above.

(b) One should then have a new equation

$$-4\pi^2 k^2 \hat{u}(k) + v^2 \hat{u}(k) = 0.$$

Solve this new equation for k.

(c) One should have two values k_1 and k_2 from the work in (b). This corresponds to the solution

$$\hat{u}(k) = \delta(k - k_1) \qquad \text{and} \qquad \hat{u}(k) = \delta(k - k_2).$$

Compute the inverse Fourier transform of the two delta functions. A linear combination of these correspond to your solution $u(t)$.

Solution 4.

(a) We have that the Fourier transformed equation is given by

$$-4\pi^2 k^2 \hat{u}(k) + v^2 \hat{u}(k) = 0,$$

since the Fourier transform converts differentiation into multiplication.

(b) Indeed, we have this equation. If we solve this equation for k, we have

$$(-4\pi^2 k^2 + v^2)\hat{u}(k) = 0,$$

which yields

$$k = \pm\frac{|v|}{2\pi}.$$

(c) Put $k_1 = -\frac{|v|}{2\pi}$ and $k_2 = \frac{|v|}{2\pi}$. Thus, we get

$$\hat{u}(k) = \delta\left(k + \frac{|v|}{2\pi}\right) \qquad \text{and} \qquad \hat{u}(k) = \delta\left(k - \frac{|v|}{2\pi}\right).$$

Then we have that

$$\mathcal{F}\left[\delta\left(k + \frac{|v|}{2\pi}\right)\right] = e^{-ivt}$$

and

$$\mathcal{F}\left[\delta\left(k - \frac{|v|}{2\pi}\right)\right] = e^{ivt}.$$

If we take a linear combination of these solutions we arrive at

$$u(t) = C_1 e^{ivt} + C_2 e^{-ivt},$$

which is the solution we have found for the harmonic oscillator equation before!