MATH 272, Homework 3

Due February 17th

Problem 1. Compute the Fourier series for the following functions on the interval $[0, L]$. Then plot your result (for $N = 1, 50, 100, 500$) compared to the original function. What do you notice if you plot the Fourier series outside the range of $[0, L]$?

(a) $f(x) = \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{-4\pi x}{L}\right)$.

(b) $\sin\left(\frac{3\pi x}{L}\right)$.

(c) $\delta(x - L/2)$.

Problem 2. Consider a function $f(x)$ that describes the height of a rubber string with rest length L. We can attach the ends of the string at $x = 0$ and $x = L$ by requiring that $f(0) = f(L) = 0$. Then, one can subject the string to an external force $g(x)$ and find the profile of the string by solving

$$-\frac{d^2}{dx^2} f(x) = g(x).$$

(a) Let $g(x) = \delta(x - L/2)$ and let $f(x)$ be given by some Fourier series. Using the equation above, solve for the coefficients of the Fourier series for $f(x)$.

(b) Plot the Fourier series for $f(x)$ for $N = 1, 5, 50$.

This is an extremely important to solve. The fact that we can determine a solution $f(x)$ where the external force is the Dirac delta function means that we have the ability to determine a the deformation of a string from a point force.

Problem 3. Compute the following Fourier transforms (using a table or WolframAlpha if need be).

(a) $\sin(3\pi x)$.

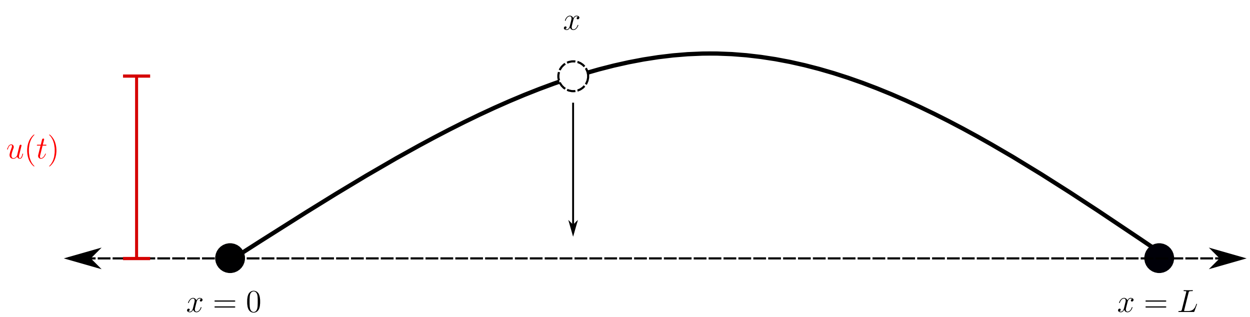
(b) $e^{\frac{-x^2}{2}}$.

(c) $\delta(x)$.

Problem 4. A common application for the Fourier transform is to solve differential equations whose domain is time $t \in [0, \infty)$. We can model how a point x on a rubber string oscillates over time consider the differential equation

$$u''(t) + v^2 u(t) = 0,$$

with initial conditions $u(0) = L$ and $u'(0) = 0$. Here $u(t)$ is the displacement of the string at position x with the initial conditions describing the string being pulled tight at time $t = 0$.



To solve this equation, we could use use methods we learned previously, or apply the Fourier transform to the whole equation by

$$\mathcal{F}(u''(t) + v^2 u(t)) = \mathcal{F}(0).$$

(a) Compute the Fourier transform above.

(b) One should then have a new equation

$$-4\pi^2 k^2 \hat{u}(k) + v^2 \hat{u}(k) = 0.$$

Solve this new equation for k.

(c) One should have two values k_1 and k_2 from the work in (b). This corresponds to the solution

$$\hat{u}(k) = \delta(k - k_1) \qquad \text{and} \qquad \hat{u}(k) = \delta(k - k_2).$$

Compute the inverse Fourier transform of the two delta functions. A linear combination of these correspond to your solution $u(t)$.