MATH 272, Homework 1

Due January 31st

Problem 1. Plot the following complex functions as vector fields. Then explain the differences between them.

(a) $f(z) = z$;

(b) $g(z) = iz$.

You can, for example, use the plotter here: `https://www.desmos.com/calculator/eijhparfmd` or find your own (Matlab for example can plot vector fields quite easily). Note that you will have to convert from the complex numbers to 2-dimensional real vectors (i.e., vectors in $\mathbb{R}^2$).

Problem 2. Let $\Psi(x)$ be a complex function with domain $[0, L]$. Show that multiplication by a global phase $e^{i\theta}$ does not affect the norm of $\Psi(x)$ under the Hermitian (integral) inner product. In more generality, this shows that you cannot fully determine a quantum state – there will always be an undetermined phase.

Problem 3. Consider the real function $f(x) = 1$ on the domain $[0, L]$.

(a) What is the norm of f, $\|f\|$?

(b) Normalize $f(x)$.

(c) Find a nonzero normalized polynomial of degree ≤ 1 that is orthogonal to $f(x)$.

Problem 4. A wavefunction $\Psi(x)$ for a particle in the 1-dimensional box $[0, L]$ could be written as a superposition of normalized states

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

That is,

$$\Psi(x) = \sum_{n=1}^{\infty} a_n \psi_n(x),$$

for some choice of the coefficients a_n.

(a) Let $a_n = \frac{\sqrt{6}}{n\pi}$. Show that $\Psi(x)$ is normalized. *Hint: first, use orthogonality of the states $\psi_n(x)$ to your advantage. Then you will need to know what an infinite series evaluates to. Use a tool like WolframAlpha to evaluate this series.*

(b) Note that we can approximate $\Psi(x)$ by taking a finite sum approximation up to some chosen N by

$$\Psi(x) \approx \sum_{n=1}^{N} a_n \psi_n(x).$$

Plot the approximation of $\Psi(x)$ for $N = 1, 5, 50, 100$. *Hint: you can modify my Desmos examples.*

Problem 5. Suppose we have two vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^3$. We can compute the distance between the vectors

$$d(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = \|\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}\| = \sqrt{(\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}}) \cdot (\vec{\boldsymbol{u}} - \vec{\boldsymbol{v}})}.$$

That is to say, we inherit not only a norm from an inner product, but a distance function from a norm! Intuitively, we are finding the length (or norm) of the vector extending from the head of $\vec{\boldsymbol{v}}$ to the head of $\vec{\boldsymbol{u}}$.

(a) Show that

$$d(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) = \sqrt{\|\vec{\boldsymbol{u}}\|^2 + \|\vec{\boldsymbol{v}}\|^2 - 2\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}.$$

(b) Compute the distance between vectors $\vec{\boldsymbol{u}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{v}} = \hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}$.

(c) Extend this notion to compute the distance between the Legendre polynomials $f_1, f_2 \colon [-1, 1] \to \mathbb{R}$ where $f_1(x) = \sqrt{\frac{3}{2}}x$ and $f_2(x) = \sqrt{\frac{5}{8}}(1 - 3x^2)$. *Hint: make sure you use the correct integral inner product for this domain!*