MATH 272, Exam 3
COVID-19 Edition
Due May 4^{th} by 11:59PM

Name ________________________________

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page, but no other online resources (including calculators or WolframAlpha). for this portion of the exam. **Do not discuss any problem any other person.** All of your solutions should be easily identifiable and supporting work must be shown. Ambiguous or illegible answers will not be counted as correct.

Problem 1	____/10	**Problem 2**	____/21
Problem 3	____/21	**Problem 4**	____/10
Problem 5	____/13		

Note, these problems span two pages.

Problem 1. Related to the wave equation is the *Korteweg-de Vries (KdV) equation.* It can be written as

$$\frac{\partial}{\partial t}u + \frac{\partial^3}{\partial x^3}u - 6u\frac{\partial}{\partial x}u = 0,$$

where $u = u(x, t)$ is a function of a single spatial variable x and one temporal variable t. This equation seeks to model waves for thin materials such as waves for shallow water surfaces or in crystal lattices.

(a) (**5 pts.**) If u and v are both solutions to this equation, is the linear combination $w = \alpha u + \beta v$ for constants α and β also a solution?

(b) (**5 pts.**) Explain why separation of variables (with the variables x and t) will not work with this equation. *Hint: if you start trying this ansatz, can you see where it fails?*

Problem 2. Consider the 1-dimensional wave equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}\right) u(x,t) = 0,$$

with domain $\Omega = [0,1]$. Without any boundary or initial conditions, the general solution to the PDE is given by

$$u(x,t) = (A\sin(\lambda x) + B\cos(\lambda x))\,(C\sin(\lambda t) + D\cos(\lambda t))\,,$$

where A, B, C, and D are undetermined constants and λ is the separation constant.

(a) (**5 pts.**) With Neumann boundary conditions $\frac{\partial}{\partial x}u(0,t) = 0 = \frac{\partial}{\partial x}u(1,t)$ and initial conditions $u(x,0) = \cos(\pi x)$ and $\frac{\partial}{\partial t}u(x,0) = 0$, find the particular solution.

(b) (**5 pts.**) We can change the problem slightly by forcing new boundary and initial conditions. Take instead the mixed conditions $u(0,t) = 0$ and $\frac{\partial}{\partial x}u(1,t) = 0$ with initial conditions $u(x,0) = \sin\left(\frac{\pi x}{2}\right)$ and $\frac{\partial}{\partial t}u(x,0) = 0$ and find the particular solution.

(c) (**3 pts.**) Describe possible physical scenarios that could would be modeled by the problems posed in (a) and (b). Pay special attention the boundary conditions.

(d) (**5 pts.**) Plot your two solutions at the times $t = 0$, $t = 1/2$, and $t = 1$.

(e) (**3 pts.**) The amount of peaks/troughs one sees in a vibrating string (made from the same material) determines the pitch of the sound it will make. More peaks/troughs correspond to higher frequency sound (hence the name frequency). Which solution, (a) or (b), has a higher frequency sound output? Why?

Problem 3. Consider the 1-dimensional heat equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) u(x,t) = 1$$

on the domain $\Omega = [0,1]$ with Dirichlet boundary conditions $u(0,t) = 1 = u(1,t)$ and initial condition $u(x,0) = \sin(\pi x) - \frac{1}{2}x^2 + \frac{1}{2}x + 1$. You can imagine this problem describes a rod made of radioactive material that is constantly generating heat while being cooled down from both ends.

(a) (**5 pts.**) First, find the equilibrium solution by solving

$$-\frac{\partial^2}{\partial x^2} u_E(x) = 1,$$

and using the above boundary conditions.

(b) (**2 pts.**) Do you expect that this rod will approach a constant temperature over a very long time? Why or why not. Explain.

(c) (**6 pts.**) Next, using separation of variables, find a general solution $v(x,t)$ that solves the source free heat equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) v(x,t) = 0,$$

that satisfies the boundary conditions $v(0,t) = 0 = v(1,t)$. *Hint: you should get a general solution for every positive integer n.*

(d) (**3 pts.**) Take $u(x,t) = v(x,t) + u_E(x)$ as your candidate solution and match the unitial condition $u(x,0)$ to find the particular solution. *Hint: you may want to think of $v(x,t)$ as a sum of the solutions you found in (c) and determine coefficients from there. You can see examples of this in the notes.*

(e) (**3 pts.**) Now, show that $u(x,t) = v(x,t) + u_E(x)$ solves the original problem.

(f) (**2 pts.**) Explain (to the best of your ability) why we can split up the solution $u(x,t)$ into the two functions $v(x,t)$ and $u_E(x)$. Thinking physically may help you understand what's going on. Thinking mathematically may allow you to see how the two functions mesh together in a useful way.

Problem 4. The time dependent Schrödinger equation for the free particle in the 1-dimensional box $\Omega = [0, L]$ is given by

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - i\hbar\frac{\partial}{\partial t}\right)\Psi(x,t) = 0,$$

with boundary conditions $\Psi(0,t) = 0 = \Psi(L,t)$. The states were given by

$$\psi_n(x,t) = e^{-i\frac{n^2\pi^2\hbar}{2mL^2}t}\sin\left(\frac{n\pi x}{L}\right).$$

(a) (**5 pts.**) Using Euler's formula, plot the real and imaginary parts of the state $\psi_1(x,t)$ for times $t = 0$, $t = 1/2$ and $t = 1$. For the simplicity of plotting, use $m = \hbar = L = 1$. However, keep these constants in the equation for the remainder of the problem.

(b) (**5 pts.**) Show that the real and imaginary parts of $\psi_1(x,t)$ solve the wave equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u(x,t) = 0,$$

for a certain value of c with Dirichlet boundary conditions $u(0,t) = 0 = u(L,t)$. Showing this will also require you to determine c!

Problem 5. Waves can also appear in higher dimensional materials. For example, the surface of water. KdV describes this one way, but we can also use the spherically symmetric wave equation

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{k^2}{\omega^2}\frac{\partial^2}{\partial t^2}\right) u(r,t) = 0,$$

that can be used to describe waves that propagate through space from a point source.

(a) (**5 pts.**) Show that $u(r,t) = \frac{1}{r}e^{i(kr \pm \omega t)}$ is a solution to this equation.

(b) (**2 pts.**) We can take a portion (real part) of this solution by letting

$$w(r,t) = \frac{1}{r}\cos(kr + \omega t).$$

Then, using the conversions

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta,$$

convert $w(r,t)$ to a function $w(x,y,t)$.

(c) (**3 pts.**) Plot the solution $w(x,y,t)$ as a surface for $k = \omega = 1$ and for $t = 0$, $t = 1/2$ and $t = 1$.

(d) (**3 pts.**) Describe what happens to $w(x,y,t)$ as we vary ω and k. Note that ω and k must be greater than zero. Plotting this may prove useful.