MATH 272, Exam 1

Take Home Portion

Due February 19$^{\text{th}}$ at the start of class

Name ________________________________

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page, but no other online resources (including calculators or WolframAlpha). for this portion of the exam. **Do not discuss any problem any other person.** All of your solutions should be easily identifiable and supporting work must be shown. Ambiguous or illegible answers will not be counted as correct. **Print out this sheet and staple your solutions to it. Use a new page for each problem.**

Problem 1 ____/15 **Problem 2** ____/10

Note, these problems span two pages.

Problem 1. Sometimes breaking down operators into smaller components can help one better understand a problem. Given this, let's consider the Hamiltonian operator $\hat{H}$ for the Quantum Harmonic Oscillator (QHO) given by

$$\hat{H} = \hat{T} + V(x) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2, \qquad \text{where} \qquad \hat{p} = -i\hbar\frac{d}{dx}$$

where m and ω are real constants. Then, the following are states of the system

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \qquad \text{and} \qquad \psi_1(x) = x\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}},$$

where $x \in \mathbb{R}$.

(a) (**3 pts.**) One can generate $\psi_1(x)$ from $\psi_0(x)$ by using the *raising operator*

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}\hat{p}\right).$$

Show that $\hat{a}^\dagger\psi_0 = \psi_1$.

(b) (**2 pts.**) Operators, just like matrices, do not always commute! So, we often want to see how "far from commuting" two operators are. To this end, let $\Psi(x)$ be an arbitrary function, then the *commutator* $[x, \hat{p}]$ is defined by

$$[x, \hat{p}]\Psi(x) := x\left(\hat{p}\left(\Psi(x)\right)\right) - \hat{p}\left(x\left(\Psi(x)\right)\right).$$

Show that $[x, \hat{p}] = i\hbar$. *Note that we also have* $[\hat{p}, x] = -[x, \hat{p}]$.

(c) (**2 pts.**) We can define the *lowering operator* $\hat{a}$ as the adjoint of $\hat{a}^\dagger$ which is given by

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}\hat{p}\right).$$

This allows us to define the *number operator* $\hat{N} = \hat{a}^\dagger\hat{a}$. Compute $\hat{N}$. *Hint: The commutator* $[x, \hat{p}]$ *appears in this equation.*

(d) (**2 pts.**) From this, we can define the Hamiltonian operator by $\hat{H} = \hbar\omega\left(N + \frac{1}{2}\right)$. Show that this is true.

(e) (**3 pts.**) Argue that $\hat{H}$ is Hermitian. *Hint: You can use results from our notes and homework.*

(f) (**3 pts.**) Show that the ground state ψ_0 is an eigenfunction of $\hat{H}$ with eigenvalue $\frac{1}{2}\hbar\omega$. (*This means that the lowest energy state of the system has positive (nonzero) energy!*)

(g) (**Bonus 2pts.**) The fact that $\hat{H}$ is Hermitian implies that ψ_0 and ψ_1 must be orthogonal with respect to the inner product

$$\langle\Psi, \Phi\rangle = \int_{-\infty}^{\infty} \Psi(x)\Phi^*(x)dx.$$

Can you argue that this is true for the given ψ_0 and ψ_1?

Problem 2. What is the Dirac delta? Let us define the function

$$S_t(x) = \begin{cases} 0 & x < -t \\ \frac{1}{2t} & -t \leq x \leq t \\ 0 & x > t \end{cases},$$

where t is some positive real number.

(a) (**2 pts.**) Let us recall the Dirac delta $\delta(x)$. Given any function $f(x)$, what is

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx?$$

(b) (**3 pts.**) Show that for any fixed $t > 0$ that

$$\int_{-\infty}^{\infty} S_t(x) dx = 1.$$

(c) (**3 pts.**) Now let $F(x)$ be the antiderivative of $f(x)$. Evaluate the integral

$$\int_{-\infty}^{\infty} S_t(x) f(x) dx.$$

Note, your answer should be in terms of the antiderivative $F(x)$ and t.

(d) (**2 pts.**) Recall that by definition, we have that $F'(x) = f(x)$. In other words,

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x - \Delta x)}{2\Delta x} = F'(x) = f(x).$$

With your answer in (c), take the limit as $t \to 0$ to show that you recover the answer you have in (a).

Visit this Desmos URL to have an interactive look at the function $S_t(x)$ for various values of t. `https://www.desmos.com/calculator/eh5jmqeky1`