MATH 272, Exam 1

Name ________________________________

Instructions No textbook, homework, calculators, phones, or smart watches may be used for this exam. The exam is designed to take 50 minutes and must be submitted at the end of the class period. All of your solutions should be easily identifiable and supporting work must be shown. You may use any part of this packet as scratch paper, but please clearly label what work you want to be considered for grading. Ambiguous or illegible answers will not be counted as correct.

Only the highest scoring five problems will be counted towards your total score. You cannot get over 75 points.

Problem 1	____/15
Problem 2	____/15
Problem 3	____/15
Problem 4	____/15
Problem 5	____/15
Problem 6	____/15
Total	_____/75

There are extra pages between each problem for scratch work.
Please circle your answers!

A table of Fourier transforms and their inverses.

$f(x)$	$\hat{f}(k)$
$\delta(x)$	1
1	$\delta(k)$
e^{iax}	$\delta\left(k - \frac{a}{2\pi}\right)$
$\cos(ax)$	$\frac{\delta\left(k-\frac{a}{2\pi}\right)+\delta\left(k+\frac{a}{2\pi}\right)}{2}$
$\sin(ax)$	$\frac{\delta\left(k-\frac{a}{2\pi}\right)-\delta\left(k+\frac{a}{2\pi}\right)}{2i}$
$e^{-\alpha x^2}$	$\sqrt{\frac{\pi}{\alpha}} e^{\frac{(\pi k)^2}{\alpha}}$

Integration by parts:

$$\int_a^b u dv = uv|_a^b - \int_a^b v du.$$

Problem 1.

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(a) The equation

$$-\frac{d^2}{dx^2} f(x) = \delta(x)$$

where $x \in \mathbb{R}$ has a solution. ☐ ☐

(b) A linear operator $\mathcal{L}$ satisfies ☐ ☐

$$\mathcal{L}(f + \alpha g) = \mathcal{L}f + \alpha \mathcal{L}g$$

for any constant α and functions f and g.

(c) A constant function $f(x) = c$ is an eigenfunction of the derivative operator $\mathcal{L} = \frac{d}{dx}$ ☐ ☐

(d) The Dirac delta defined on $[0, L]$ cannot be written as a Fourier series. ☐ ☐

(e) An inner product $\langle \cdot, \cdot \rangle$ also allows one to define a distance function $d(\cdot, \cdot)$. ☐ ☐

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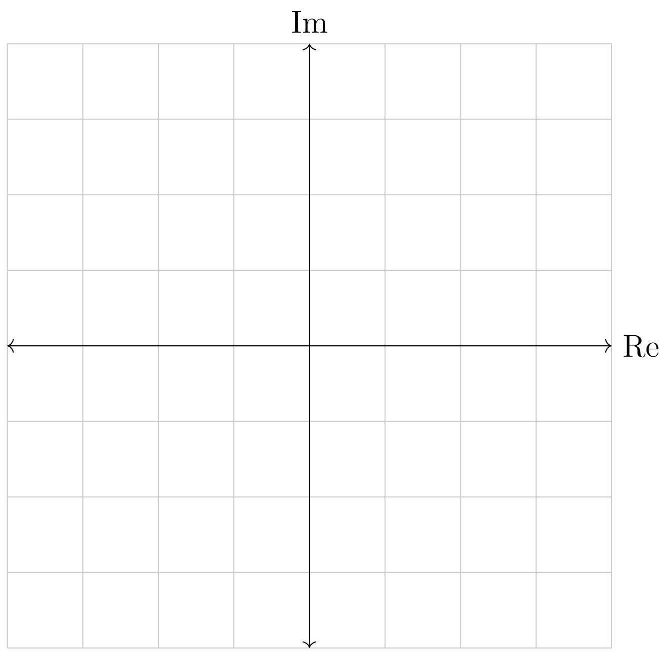
Problem 2. Consider the complex valued functions defined on the domain $[-\pi, \pi]$,

$$f(x) = x + i\sin(x) \qquad \text{and} \quad g(x) = a + bi$$

where a and b are real constants.

(a) (**5 pts.**) Plot the function f in the complex plane below.



(b) (**5 pts.**) Compute the norm of the function $\|g\|$ using the Hermitian inner product for the interval $[-\pi, \pi]$.

(c) (**5 pts.**) Show that $g(x)$ is orthogonal to $f(x)$. *Hint: Use the fact that* $g(x)$ *is constant and* $f(x)$ *is odd.*

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Problem 3. Consider the linear differential equation $-\frac{d^2}{dx^2}f(x) = \omega^2 f(x)$ where ω is a constant. In this problem we seek to understand the premise behind the Fourier transform.

(a) (**4 pts.**) Suppose that both $\chi_1(x)$ and $\chi_2(x)$ are solutions. Show that $f(x) = \alpha_1\chi_1(x) + \alpha_2\chi_2(x)$ is also a solution.

(b) (**4 pts.**) If we have that $\chi_n(x)$ is a solution for all integers n, show or explain why

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n \chi_n(x),$$

is also a solution.

(c) (**4 pts.**) If indeed $\chi_k(x)$ is a solution for all real numbers k, show or explain why

$$f(x) = \int_{-\infty}^{\infty} \alpha_k \chi_k(x) dk,$$

is also a solution.

(d) (**3 pts.**) If every function $f(x)$ corresponds to a unique list of coefficients α_k, explain why it suffices to manipulate the coefficients α_k to solve a differential equation.

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Problem 4. For the following, assume we are using the Hermitian inner product

$$\langle f, g\rangle = \int_0^L f(x)g^*(x)dx.$$

Also, assume any functions satisfy $f(0) = f(L)$.

(a) (**4 pts.**) Let $\hat{a}$ be an operator and $\hat{a}^\dagger$ be the adjoint. Show that the operator $\hat{a}\hat{a}^\dagger$ is Hermitian.

(b) (**3 pts.**) Let $\hat{a} = \frac{d}{dx}$. Show using integration by parts that the adjoint operator is $\hat{a}^\dagger = -\frac{d}{dx}$.

(c) (**4 pts.**) Is $\hat{a}$ Hermitian? Explain.

(d) (**4 pts.**) Could there be a solution to the equation

$$-\frac{d^2}{dx^2}f(x) = if(x)?$$

Explain.

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Problem 5. Let $f(x)$ be a function defined on $[0, L]$. We want to represent $f(x)$ as a Fourier series by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right).$$

(a) (**4 pts.**) How do you compute the coefficient a_0?

(b) (**4 pts.**) How do you compute the coefficients a_n and b_n?

(c) (**4 pts.**) Explain why the function $f(x) = \frac{1}{\left|x - \frac{L}{2}\right|}$ does not have a Fourier series representation. *Hint: Can you compute the coefficients?*

(d) (**3 pts.**) If a function has a Fourier series, explain why that series representation must be unique. *Hint: Could you possibly get two different answers for the coefficients?*

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Problem 6. The Fourier transform of a function $f(x)$ is given by

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i2\pi kx}dx = \hat{f}(k).$$

Also, note that

$$\mathcal{F}\left[\frac{d}{dx}f(x)\right] = (i2\pi k)\hat{f}(k),$$

which is proven using integration by parts.

(a) (**5 pts.**) Make a quick argument on why

$$\mathcal{F}\left[\frac{d^n}{dx^n}f(x)\right] = (i2\pi k)^n\hat{f}(k).$$

given the statements above.

(b) (**5 pts.**) Using previous results, apply the Fourier transform to the following differential equation.

$$\frac{d^2}{dx^2}f(x) + f(x) = \sin(\omega x).$$

Then solve for $\hat{f}(k)$ in terms of k.

(c) (**5 pts.**) Describe how we could find the solution $f(x)$ from what we found in (b).

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