MATH 271, Homework 9, *Solutions*

Due November 15^{TH}

Problem 1. Compute the following:

(a)

$$[A] = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

(b)

$$[B] = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}$$

(c) Take

$$[M] = \begin{pmatrix} 10 & 15 \\ 20 & 10 \end{pmatrix}$$

and

$$[N] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Compute $[M][N]$ and $[N][M]$ to see that matrices do not commute in general.

Solution 1.

(a) Since we have a 1×3-matrix multiplied with a 3×1-matrix, we know that $[A]$ should be a 1×1-matrix.

$$\begin{aligned} [A] &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \\ &= (1 \cdot 2 + 1 \cdot 1 + 1 \cdot 3) \\ &= (6). \end{aligned}$$

(b) Here, we should expect that $[B]$ is a 3×2-matrix.

$$[B] = \begin{pmatrix} 24 & 26 \\ 64 & 66 \\ 104 & 106 \end{pmatrix}.$$

(c) Here $[M]$ and $[N]$ are square, so multiplying will give us the same shape matrix. We have

$$[M][N] = \begin{pmatrix} 40 & 35 \\ 40 & 50 \end{pmatrix},$$

as well as

$$[N][M] = \begin{pmatrix} 50 & 35 \\ 40 & 40 \end{pmatrix}.$$

From this we can see that $[M][N] \neq [N][M]$ in general!

Problem 2. A linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

$$[T] = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

(a) Compute how T transforms the standard basis elements for $\mathbb{R}^3$. That is, find

$$T(\hat{\boldsymbol{e}}_1), \qquad T(\hat{\boldsymbol{e}}_2), \qquad T(\hat{\boldsymbol{e}}_3)$$

and relate these values to the columns of $[T]$.

(b) Is the transformed basis $T(\hat{\boldsymbol{e}}_1)$, $T(\hat{\boldsymbol{e}}_2)$, and $T(\hat{\boldsymbol{e}}_3)$ linearly independent? Do these vectors form a basis for $\mathbb{R}^3$?

(c) If we apply this linear transformation to the unit cube (that is, all points who have (x, y, z) coordinates with $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$), what will the volume of the transformed cube be? (*Hint: use the determinant.*)

Solution 2.

(a) The point here is that we can understand the matrix $[T]$ and matrix multiplication better by seeing how the basis vectors are transformed. So we have

$$\begin{aligned} T(\hat{\boldsymbol{e}}_1) &= \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\ &= \hat{\boldsymbol{e}}_1 + 2\hat{\boldsymbol{e}}_2, \end{aligned}$$

which is just the first column of the matrix. Then we have

$$\begin{aligned} T(\hat{\boldsymbol{e}}_2) &= \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \\ &= 2\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 + 2\hat{\boldsymbol{e}}_3, \end{aligned}$$

which is just the second column of the matrix. Lastly we have

$$\begin{aligned} T(\hat{\boldsymbol{e}}_3) &= \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \\ &= 2\hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3, \end{aligned}$$

which is the last column of the matrix.

(b) Yes. You can see this in the following ways: By row reducing to the identity matrix, by showing that the kernel is trivial, or by computing the determinant and showing it is nonzero. Let us use the determinant.

$$\det[T] = -7.$$

therefore the columns are linearly independent. Since the columns are exactly $T(\hat{\boldsymbol{e}}_j)$ by definition, those vectors are independent. Since it is a set of 3 independent vectors in $\mathbb{R}^3$, it is a basis and all of $\mathbb{R}^3$ is in the span of those transformed vectors.

(c) The three basis vectors

$$\hat{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

define the volume of the unit cube. That is, the parallelepiped generated by $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ is the unit cube. Hence, if we know how these vectors are transformed, we just need to find the volume of the paralellepiped given by the transformed vectors $T(\hat{\boldsymbol{e}}_1)$, $T(\hat{\boldsymbol{e}}_2)$, and $T(\hat{\boldsymbol{e}}_3)$. Now, we can collect these vectors into a matrix,

$$\begin{pmatrix} | & | & | \\ T(\hat{\boldsymbol{e}}_1) & T(\hat{\boldsymbol{e}}_2) & T(\hat{\boldsymbol{e}}_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},$$

which is exactly $[T]$ and this should not be shocking since this is how we defined a matrix representation in the first place. Now, the determinant of the matrix gives us the signed volume of the parallelepiped generated by the three column vectors, and hence

$$\text{Area} = |\det[T]| = |-7| = 7.$$

Problem 3.

(a) Show that for any 2×2-matrix that the sign of the determinant changes if either a row or column is swapped. *Note: this is true for square matrices of any size.*

(b) Show that for any 2×2-matrix that multiplying a column by a constant scales the determinant by that constant as well. *Note: this is true for square matrices of any size.*

(c) Show that for any 2×2-matrix that adding a scalar multiple one column to the other will not change the determinant. *Note: this is true in broader generality. In fact, adding linear combinations of columns to another column will not change the determinant.*

(d) Using these facts, argue that a square matrix with columns that are linearly dependent must have a determinant of zero.

Solution 3.

(a) Let

$$[A] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an arbitrary 2×2-matrix. Then we have

$$\det([A]) = ad - bc.$$

Now, if we swap rows we have

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc).$$

Now, we can do the same with columns to get

$$\begin{vmatrix} b & a \\ d & c \end{vmatrix} = bc - ad = -(ad - bc).$$

(b) Let us compute the determinant of

$$\begin{vmatrix} \alpha a & b \\ \alpha c & d \end{vmatrix} = \alpha ad - \alpha bc = \alpha(ad - bc).$$

Similarly, we will have the same if we scale the other column. In fact, this is true for rows as well.

(c) Let us add the first column to the second. We get

$$\begin{vmatrix} \alpha a & \alpha a + b \\ \alpha c & \alpha c + d \end{vmatrix} = a(\alpha c + d) - (\alpha a + b)c = \alpha ac + ad - \alpha ac - bc = ad - bc.$$

The same will be true if we add a scalar copy of column 2 to column 1.

(d) Let us just show this for a 3×3-matrix as the argument is the same for the most general case. Let

$$[A] = \begin{pmatrix} | & | & | \\ \vec{\boldsymbol{A}}_1 & \vec{\boldsymbol{A}}_2 & \vec{\boldsymbol{A}}_3 \\ | & | & | \end{pmatrix}.$$

Then if the columns of $[A]$ are linearly dependent, we have that

$$\alpha_1 \vec{\boldsymbol{A}}_1 + \alpha_2 \vec{\boldsymbol{A}}_2 + \alpha_3 \vec{\boldsymbol{A}}_3 = \vec{\boldsymbol{0}}$$

with at least one $\alpha_i \neq 0$. Specifically, this means that one vector can be written as a linear combination of the others. That is we can take

$$\vec{\boldsymbol{A}}_3 = \frac{-1}{\alpha_3}(\alpha_1 \vec{\boldsymbol{A}}_1 + \alpha_2 \vec{\boldsymbol{A}}_2),$$

so long as $\alpha_3 \neq 0$. If $\alpha_3 = 0$, then choose another vector to write as a linear combination of the others. Then, we can subtract the quantity

$$\frac{-1}{\alpha_3}(\alpha_1 \vec{\boldsymbol{A}}_1 + \alpha_2 \vec{\boldsymbol{A}}_2)$$

from column 3 in $[A]$ to get

$$\begin{pmatrix} | & | & | \\ \vec{\boldsymbol{A}}_1 & \vec{\boldsymbol{A}}_2 & \vec{\boldsymbol{0}} \\ | & | & | \end{pmatrix},$$

which has a determinant of zero. Since we only added a linear combination of columns to another column, this did not change the determinant and hence we must have $\det([A]) = 0$. I will leave it open for you to do this for an $n \times n$-matrix.

Problem 4. Consider the equation

$$[A]\vec{\boldsymbol{v}} = \vec{\mathbf{0}},$$

where

$$[A] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Are the columns of $[A]$ linearly independent or dependent? Explain.

(b) What vector(s) $\vec{\boldsymbol{v}}$ satisfy this equation? In other words, what is $\text{Null}([A])$?

(c) Using what you found above, what must $\det([A])$ be equal to? *Hint: you do not need to compute the determinant!*

Solution 4.

(a) The columns are dependent as the leftmost column is equal to the rightmost column.

(b) To solve the homogeneous equation we take

$$[M] = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

Then we can subtract row one from row three to get

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which corresponds to the equations

$$\begin{aligned} 0x + y + 0z &= 0 \\ x + 0y + z &= 0 \\ 0x + 0y + 0z &= 0. \end{aligned}$$

Hence we have that $z = -x$ and $y = 0$. Thus any vector of the form

$$\vec{\boldsymbol{v}} = \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix}$$

for any $t \in \mathbb{R}$ is a solution to this equation. In other words, the set described above is $\ker[A]$.

(c) The determinant must be equal to zero since $\ker[A]$ is nontrivial (i.e., it contains more than just the zero vector). One can also note the columns are dependent which implies this as well. This goes to show a bit on how these ideas are all connected.

Problem 5. Compute the following.

(a)

$$\det[A] = \begin{vmatrix} -3 & 1 & 5 \\ -3 & 4 & 2 \\ -3 & 2 & 1 \end{vmatrix}$$

(b)

$$\det[B] = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(c) Compute $\det([A][B])$ using properties of the determinant. *Hint: this should be very quick to do. Do not compute the product of the matrices $[A]$ and $[B]$!*

(d) Compute $\operatorname{tr}([C])$ and $\operatorname{tr}([D])$ where

$$[C] = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -2 & -2 & 0 \end{pmatrix} \qquad \text{and} \qquad [D] = \begin{pmatrix} -3 & 1 & 1 \\ 2 & -2 & 4 \\ -1 & -1 & -1 \end{pmatrix}.$$

(e) Compute $\operatorname{tr}([C][D])$ and compare it to $\operatorname{tr}([D][C])$.

Solution 5.

(a) We can expand along any row or column and in this case, there are no zeros to make the computation quicker. So we have

$$\begin{aligned} \begin{vmatrix} -3 & 1 & 5 \\ -3 & 4 & 2 \\ -3 & 2 & 1 \end{vmatrix} &= -3\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -3 & 2 \\ -3 & 1 \end{vmatrix} + 5\begin{vmatrix} -3 & 4 \\ -3 & 2 \end{vmatrix} \\ &= -3(4-4) - 1(-3+6) + 5(-6+12) \\ &= 27. \end{aligned}$$

(b) Similarly, we get

$$\det([B]) = 0.$$

(c) We know that $\det([A][B]) = \det([A])\det([B])$ and thus we have that $\det([A][B]) = 0$.

(d) The trace is the sum of the diagonal entries. Thus we have

$$\begin{aligned} \operatorname{tr}[C] &= 1 + 1 + 0 = 2, \\ \operatorname{tr}[D] &= -3 - 2 - 1 = -6. \end{aligned}$$

(e) Then we can compute $[C][D]$,

$$[C][D] = \begin{pmatrix} -5 & -1 & -1 \\ -7 & -3 & 3 \\ 2 & 2 & -10 \end{pmatrix}.$$

Hence we have

$$\mathrm{tr}([C][D]) = -18.$$

Note that under cyclic permutations, the trace is invariant, hence

$$\mathrm{tr}([C][D]) = \mathrm{tr}([D][C])$$

even though $[C][D] \neq [D][C]$

Problem 6. Consider some linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$. Let $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k$ be vectors in $\mathrm{Null}(T)$.

(a) Show that the span of these vectors is also in the kernel of T.

(b) How many linearly independent vectors can be in the kernel? Give bounds using n and m.

Solution 6.

(a) An arbitrary vector $\vec{\boldsymbol{v}}$ in the span of $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k$ is given by

$$\vec{\boldsymbol{v}} = \alpha_1\vec{\boldsymbol{v}}_1 + \alpha_2\vec{\boldsymbol{v}}_2 + \cdots + \alpha_k\vec{\boldsymbol{v}}_k.$$

Since $\vec{\boldsymbol{v}}$ is arbitrary, if we show $\vec{\boldsymbol{v}} \in \ker T$, then we are done. So we take

$$\begin{aligned} T(\vec{\boldsymbol{v}}) &= T(\alpha_1\vec{\boldsymbol{v}}_1 + \alpha_2\vec{\boldsymbol{v}}_2 + \cdots + \alpha_k\vec{\boldsymbol{v}}_k) \\ &= \alpha_1 T(\vec{\boldsymbol{v}}_1) + \alpha_2 T(\vec{\boldsymbol{v}}_2) + \cdots + \alpha_k T(\vec{\boldsymbol{v}}_k) && \text{by linearity of } T \\ &= 0 && \text{since } T(\vec{\boldsymbol{v}}_i) = 0 \text{ for all } i = 1, \ldots, k. \end{aligned}$$

Thus the span of $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k$ is also in the nullspace of T.

(b) With $T\colon \mathbb{R}^n \to \mathbb{R}^m$ we can take the transformation $T(\vec{\boldsymbol{v}}) = \vec{\boldsymbol{0}}$ for every vector $\vec{\boldsymbol{v}} \in \mathbb{R}^n$. Note that this transformation always exists and will always have the largest kernel. Thus, the kernel of T would have as many as n-linearly independent vectors since there can be at most n-linearly independent vectors in $\mathbb{R}^n$. This fact is independent of the value for m.

Taking m into account now, if $m < n$, then we must have $n - m$ vectors in the kernel of T at the very least. The argument is somewhat geometrical as T removes $n - m$ dimensions in the process and as such, we remove $n - m$ linearly independent vectors (as those vectors span those removed dimensions). Thus, if $m < n$ we have that

$$n - m \leq \text{number of L.I. vectors in kernel of } T \leq n.$$

In the case $m \geq n$ we have

$$0 \leq \text{number of L.I. vectors in kernel of } T \leq n.$$

To see why this is true, we let $\vec{\boldsymbol{v}} = v_1\hat{\boldsymbol{e}}_1 + v_2\hat{\boldsymbol{e}}_2 + \cdots + v_n\hat{\boldsymbol{e}}_n$ and note that if $m \geq n$ we can take

$$T(\vec{\boldsymbol{v}}) = v_1\hat{\boldsymbol{e}}_1 + v_2\hat{\boldsymbol{e}}_2 + \cdots + v_n\hat{\boldsymbol{e}}_n + 0\hat{\boldsymbol{e}}_{n+1} + \cdots + 0\hat{\boldsymbol{e}}_m,$$

which shows that there are no nontrivial vectors in the kernel of this T.

Problem 7. Suppose that the operator $T\colon V \to V$ has a nonzero kernel (e.g., some vector $\vec{\boldsymbol{v}}$ other than $\vec{\mathbf{0}}$ is in the kernel). Prove that T has no inverse. *Hint: this means you can construct a vector that is not in the image of T!*

Solution 7. I will give two proofs for this:

- It is true that
$$\dim V = \dim \ker T + \dim \operatorname{im} T.$$
Therefore if $\dim \ker T > 0$, then $\dim \operatorname{im} T < \dim V$. Hence, there exists at least one nonzero vector $\vec{\boldsymbol{v}} \in V$ that is not in the image of T. Therefore, the is no $\vec{\boldsymbol{u}} \in V$ such that $T(\vec{\boldsymbol{u}}) = \vec{\boldsymbol{v}}$ and so T has no inverse.

- Since T has a nonzero kernel, take $\vec{\boldsymbol{v}} \neq \vec{\mathbf{0}} \in \ker T$. Then $T(\vec{\boldsymbol{v}}) = \vec{\mathbf{0}} = T(\vec{\mathbf{0}})$ Therefore if T^{-1} did exist, it must be that $T^{-1}(\vec{\mathbf{0}}) = \vec{\boldsymbol{v}}$ and $T^{-1}(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$. This is a contradiction so the supposition must be false.

Problem 8. The previous problem will be very helpful for these two parts.

(a) Let $T\colon V \to V$ be an operator such that $\det[T] = 0$. Explain why there exists a solution to the homogeneous equation $T\vec{\boldsymbol{u}} = \vec{\mathbf{0}}$.

(b) Suppose $S\colon V \to V$ is another operator such that $\det[S] \neq 0$. Explain why there exists a solution to the inhomogeneous equation $S\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}$ for any $\vec{\boldsymbol{w}} \in V$.

Solution 8.

(a) Since $\det[T] = 0$ it must be that the kernel of T is nonzero. You can see this fact in many ways. For instance, the columns of $[T]$ are linearly dependent and hence you can take a linear combination of $T(\vec{\boldsymbol{e}}_j)$ and get the zero vector, for instance

$$u_1 T(\vec{\boldsymbol{e}}_1) + \cdots + u_n T(\vec{\boldsymbol{e}}_n) = \vec{\mathbf{0}}$$

where not all u_j are zero. Hence, there exists a vector $\vec{\boldsymbol{u}} = \sum_{j=1}^n u_j \vec{\boldsymbol{e}}_j \in \ker T$ and by definition/construction $T\vec{\boldsymbol{u}} = \vec{\mathbf{0}}$.

(b) Since $\det[S] \neq 0$ then the columns of S are linearly independent. Since there are n-linearly independent vectors in an n-dimensional space (assuming V is dimension n), they form a basis and span V. Note that for $\vec{\boldsymbol{v}} = \sum_{j=1}^n v_j \hat{\boldsymbol{e}}_j$ and

$$S\vec{\boldsymbol{v}} = \sum_{j=1}^{n} v_j S(\hat{\boldsymbol{e}}_j)$$

which is just a linear combination of the columns of $[S]$. Since the columns of $[S]$ are a basis, for any vector $\vec{\boldsymbol{w}} \in V$, we have $\vec{\boldsymbol{w}} \in \operatorname{Span}\{S(\hat{\boldsymbol{e}}_1), \ldots, S(\hat{\boldsymbol{e}}_n)\}$ which is exactly what $S\vec{\boldsymbol{v}}$ dictates.

Problem 9. Prove that the eigenvectors with eigenvalue 0 of an operator $T\colon V \to V$ correspond to vectors in the kernel of T.

Solution 9. Let $\vec{\boldsymbol{v}} \in V$ be an eigenvector with eigenvalue $\lambda = 0$. Then

$$T\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}} = 0\vec{\boldsymbol{v}} = 0.$$

Thus $\vec{\boldsymbol{v}} \in \ker T$.

Problem 10. Consider the linear operator $J \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$J(\hat{\boldsymbol{e}}_1) = \hat{\boldsymbol{e}}_2 \qquad \text{and} \qquad J(\hat{\boldsymbol{e}}_2) = -\hat{\boldsymbol{e}}_1.$$

(a) Show that operator polynomial

$$P(J) := J^2 + I \colon \mathbb{R}^2 \to \mathbb{R}^2$$

annihilates $\mathbb{R}^2$. Or, said another way, show that every $\vec{\boldsymbol{v}} \in \mathbb{R}^2$ is in the kernel of $P(J)$.

(b) Show that the characteristic polynomial of J is

$$p(\lambda) = \lambda^2 + 1.$$

Does this coincide with $P(J)$? *If need be, use your matrix representation* $[J]$ *from the previous homework.*

(c) Compute the eigenvalues λ_1 and λ_2 of J.

(d) Compute the corresponding eigenvectors of J.

(e) If we don't allow for complex scalars, J has no eigenvalues. However, J^2 does have only real eigenvalues. Using (a), show that J has eigenvalue $\lambda = -1$ with eigenvectors $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$.

(f) (Bonus) Can you argue that any nonzero rotation of $\mathbb{R}^2$ must have imaginary eigenvalues?

Solution 10. (a) Let $\vec{\boldsymbol{v}} \in \mathbb{R}^2$ be given by $\vec{\boldsymbol{v}} = v_1\hat{\boldsymbol{e}}_1 + v_2\hat{\boldsymbol{e}}_2$. Then

$$\begin{aligned}
(J^2 + I)\vec{\boldsymbol{v}} &= J^2\vec{\boldsymbol{v}} + I\vec{\boldsymbol{v}} \\
&= J(J(v_1\hat{\boldsymbol{e}}_1 + v_2\hat{\boldsymbol{e}}_2)) \\
&= J(-v_2\hat{\boldsymbol{e}}_1 + v_1\hat{\boldsymbol{e}}_2) + \vec{\boldsymbol{v}} \\
&= -v_1\hat{\boldsymbol{e}}_1 - v_2\hat{\boldsymbol{e}}_2 + \vec{\boldsymbol{v}} \\
&= \vec{\boldsymbol{0}}.
\end{aligned}$$

(b) We can form a matrix

$$[J] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\det([J] - \lambda[I]) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1.$$

(c) The eigenvalues are the roots to the characteristic polynomial so

$$\lambda^2 + 1 = 0.$$

The roots are $\lambda_1 = i$ and $\lambda_2 = -i$.

(d) For $\lambda_1 = i$, we take $([J] - i[I])\vec{\boldsymbol{p}}_1 = \vec{\mathbf{0}}$ where $\vec{\boldsymbol{p}}_1$ is the first eigenvector

$$([J] - i[I])\vec{\boldsymbol{p}}_1 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} -ip_{11} - p_{21} &= 0 \\ p_{11} - ip_{21} &= 0. \end{aligned}$$

Multiplying the bottom equation by i yields

$$ip_{11} + p_{21} = 0$$

which can be added to the first equation to cancel it off. Hence

$$p_{11} = ip_{21}.$$

So just choose $p_{21} = 1$ and

$$\vec{\boldsymbol{p}}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Similar work for λ_2 shows that a corresponding eigenvector is $\vec{\boldsymbol{p}}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Remark 1. The matrix $[P] = [\vec{\boldsymbol{p}}_1 \; \vec{\boldsymbol{p}}_2]$ diagonalizes $[J]$.

(e) Just take

$$J^2(\hat{\boldsymbol{e}}_1) = J(J(\hat{\boldsymbol{e}}_1)) = J(\hat{\boldsymbol{e}}_2) = -\hat{\boldsymbol{e}}_1$$

and

$$J^2(\hat{\boldsymbol{e}}_2) = J(J(\hat{\boldsymbol{e}}_2)) = J(-\hat{\boldsymbol{e}}_1) = -J(\hat{\boldsymbol{e}}_1) = -\hat{\boldsymbol{e}}_2$$

which shows both are eigenvectors with eigenvalue 1.

(f) An arbitrary rotation of some plane is given by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some choice of θ then the characteristic polynomial is

$$\lambda^2 - 2\lambda\cos\theta + 1$$

which has roots

$$\begin{aligned} \lambda_1 &= \cos\theta - i\sin\theta \\ \lambda_2 &= \cos\theta + i\sin\theta. \end{aligned}$$

and the same eigenvectors as $[J]$.

Problem 11. For this problem, we will consider eigenvectors of three operators that act on the space of analytic functions $C^\omega(\mathbb{C})$. Your goal should be to realize that these correspond to differential equations you have seen before.

(a) Take the operator $\frac{d}{dx}\colon C^\omega(\mathbb{C}) \to C^\omega(\mathbb{C})$. Show that the exponential $e^{kx} \in C^\omega(\mathbb{C})$ is an eigenvector (or eigenfunction) with eigenvalue k. Write down the corresponding ODE. *Hint: just by doing the problem properly, you will probably write down the ODE.*

(b) Take the operator $\frac{d^2}{dx^2}\colon C^\omega(\mathbb{C}) \to C^\omega(\mathbb{C})$. Show that there are two eigenfunctions $e^{i\omega x}$ and $e^{-i\omega x}$ with eigenvalue $-\omega^2$. Write down the corresponding ODE. *Hint: just by doing the problem properly, you will probably write down the ODE.*

(c) Take the operator $x\frac{d}{dx}\colon C^\omega(\mathbb{C}) \to C^\omega(\mathbb{C})$. Find the eigenfunctions to this operator using the fact that this corresponds to a separable ODE.

Solution 11.

(a) An eigenfunction of $\frac{d}{dx}$ is a function f such that

$$\frac{d}{dx}f = kf.$$

So, take $f = e^{kx}$ then

$$\frac{d}{dx}e^{kx} = ke^{kx}$$

is an eigenfunction.

(b) An eigenfunction of $\frac{d^2}{dx^2}$ is a function f such that

$$\frac{d^2}{dx^2}f = \lambda f = -\omega^2 f$$

where I'm taking the liberty of using $-\omega^2$ as an eigenvalue since I already know the work here. Take $f_\pm = e^{\pm i\omega x}$ then

$$\frac{d}{dx}e^{\pm i\omega x} = -\omega^2 e^{\pm i\omega x}$$

is an eigenfunction.

(c) An eigenfunction of $\frac{d}{dx}$ is a function f such that

$$x\frac{d}{dx}f = \lambda f.$$

Then

$$\begin{aligned} & x\frac{d}{dx}f = \lambda f \\ \iff \quad & \frac{1}{f}\frac{df}{dx} = \lambda\frac{1}{x} \\ \iff & \int \frac{1}{f}df = \lambda \int \frac{1}{x}dx \\ \iff & \ln f = \lambda \ln x + c \\ \iff & f = cx^{\lambda}. \end{aligned}$$

Hence the eigenfunctions are x^{λ}.

Remark 2. On polynomials, the basis functions x^j for $j = 0, \ldots, n$ are eigenvectors with eigenvalue j. So, in matrix notation for example take $P_3(\mathbb{C})$,

$$\left[x\frac{d}{dx}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

If you'd like, you can see that x acts on $P_3(\mathbb{C})$ as a *right shift operator*.

Problem 12. Consider the Legendre polynomials

$$B_L = \left\{ f_0 = \sqrt{\frac{1}{2}},\ f_1 = \sqrt{\frac{3}{2}}x,\ f_2 = \sqrt{\frac{5}{8}}(1-3x^2),\ f_3 = \sqrt{\frac{63}{8}}\left(x - \frac{5x^3}{3}\right) \right\}$$

which form a basis for $P_3(\mathbb{C})$.

(a) For polynomials $f, g \in P_3(\mathbb{C})$, define an inner product

$$\langle g, h\rangle := \int_{-1}^{1} gh^* dx.$$

Show (or find in the text or previous homeworks) evidence that the basis B_L is orthonormal with respect to this inner product.

(b) Consider the operator

$$\mathcal{L} := (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} : P_3(\mathbb{C}) \to P_3(\mathbb{C}).$$

Show that $\mathcal{L}$ is linear.

(c) Show that each Legendre polynomial f_i is an eigenvector (or eigenfunction) of $\mathcal{L}$. What are the corresponding eigenvalues? How do these eigenvalues correspond to the m that appears in Legendre's equation (see the section in our text).

Solution 12.

(a) See Homework 6 Problem 4 solution.

(b) First we know that $\frac{d}{dx}$ is linear by previous homework. The composition of linear transformations are also linear transformations so $\frac{d^2}{dx^2}$ is linear. Next, taking $f \in P_3$ we can write

$$x\frac{d}{dx} = x(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_1 x + 2\alpha_2 x^2 + 3\alpha_3 x^3$$

and also

$$x^2 \frac{d^2}{dx^2} f = 2\alpha_2 x^2 + 6\alpha_3 x^3$$

so we see both of those operators are linear as well. Hence since linear combinations of linear transformation are linear, $\mathcal{L}$ must be linear.

(c) First, since f_0 is constant and $\mathcal{L}$ acts by differentiation at least once,

$$\mathcal{L} f_0 = 0,$$

so f_0 is an eigenfunction with eigenvalue 0. Next,

$$\begin{aligned}\mathcal{L} f_1 &= (1-x^2)\frac{d^2}{dx^2}\left(\sqrt{\frac{3}{2}}x\right) - 2x\frac{d}{dx}\left(\sqrt{\frac{3}{2}}x\right) \\ &= 2\sqrt{\frac{3}{2}}x\end{aligned}$$

So f_1 is an eigenfunction with eigenvalue $1(1+1) = 2$. Next,

$$\begin{aligned}\mathcal{L}f_2 &= (1-x^2)\frac{d^2}{dx^2}\left(\sqrt{\frac{5}{8}}(1-3x^2)\right) - 2x\frac{d}{dx}\left(\sqrt{\frac{5}{8}}(1-3x^2)\right) \\ &= 6\left(\sqrt{\frac{5}{8}}(1-3x^2)\right)\end{aligned}$$

So f_2 is an eigenfunction with eigenvalue $2(2+1) = 6$. Finally,

$$\begin{aligned}\mathcal{L}f_3 &= (1-x^2)\frac{d^2}{dx^2}\left(\sqrt{\frac{63}{8}}\left(x-\frac{5x^3}{3}\right)\right) - 2x\frac{d}{dx}\left(\sqrt{\frac{63}{8}}\left(x-\frac{5x^3}{3}\right)\right) \\ &= 12\left(\sqrt{\frac{63}{8}}\left(x-\frac{5x^3}{3}\right)\right)\end{aligned}$$

So f_3 is an eigenfunction with eigenvalu $3(3+1) = 12$. So the subscript j for f_j corresponds to the α in Homework 6 Problem 4 or the m in the text.