MATH 271, Homework 9

Due December 3rd

Problem 1. Compute the following:

(a)

$$[A] = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

(b)

$$[B] = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}$$

(c) Take

$$[M] = \begin{pmatrix} 10 & 15 \\ 20 & 10 \end{pmatrix}$$

and

$$[N] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Compute $[M][N]$ and $[N][M]$ to see that matrices do not commute in general.

Problem 2. A linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

$$[T] = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

(a) Compute how T transforms the standard basis elements for $\mathbb{R}^3$. That is, find

$$T(\hat{\boldsymbol{e}}_1), \qquad T(\hat{\boldsymbol{e}}_2), \qquad T(\hat{\boldsymbol{e}}_3)$$

and relate these values to the columns of $[T]$.

(b) Is the transformed basis $T(\hat{\boldsymbol{e}}_1)$, $T(\hat{\boldsymbol{e}}_2)$, and $T(\hat{\boldsymbol{e}}_3)$ linearly independent? Do these vectors form a basis for $\mathbb{R}^3$?

(c) If we apply this linear transformation to the unit cube (that is, all points who have (x, y, z) coordinates with $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$), what will the volume of the transformed cube be? (*Hint: use the determinant.*)

Problem 3.

(a) Show that for any 2×2-matrix that the sign of the determinant changes if either a row or column is swapped. *Note: this is true for square matrices of any size.*

(b) Show that for any 2×2-matrix that multiplying a column by a constant scales the determinant by that constant as well. *Note: this is true for square matrices of any size.*

(c) Show that for any 2×2-matrix that adding a scalar multiple one column to the other will not change the determinant. *Note: this is true in broader generality. In fact, adding linear combinations of columns to another column will not change the determinant.*

(d) Using these facts, argue why a square matrix with columns that are linearly dependent must have a determinant of zero.

Problem 4. Consider the equation

$$[A]\vec{\boldsymbol{v}} = \vec{\mathbf{0}},$$

where

$$[A] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Are the columns of $[A]$ linearly independent or dependent? Explain.

(b) What vector(s) $\vec{\boldsymbol{v}}$ satisfy this equation? In other words, what is $\ker[A]$?

(c) Using what you found above, what must $\det[A]$ be equal to? *Hint: you do not need to compute the determinant!*

Problem 5. Compute the following.

(a)

$$\det[A] = \begin{vmatrix} -3 & 1 & 5 \\ -3 & 4 & 2 \\ -3 & 2 & 1 \end{vmatrix}$$

(b)

$$\det[B] = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(c) Compute $\det([A][B])$ using properties of the determinant. *Hint: this should be very quick to do. Do not compute the product of the matrices $[A]$ and $[B]$!*

(d) Compute $\operatorname{tr}([C])$ and $\operatorname{tr}([D])$ where

$$[C] = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -2 & -2 & 0 \end{pmatrix} \quad \text{and} \quad [D] = \begin{pmatrix} -3 & 1 & 1 \\ 2 & -2 & 4 \\ -1 & -1 & -1 \end{pmatrix}.$$

(e) Compute $\operatorname{tr}([C][D])$ and compare it to $\operatorname{tr}([C][D])$.

Problem 6. Consider some linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$. Let $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k$ be vectors in the kernel $\ker(T)$.

(a) Show that the span of these vectors is also in the kernel of T.

(b) How many linearly independent vectors can be in the kernel? Give bounds using n and m.

Problem 7. Suppose that the operator $T\colon V \to V$ has a nonzero kernel (e.g., some vector $\vec{\boldsymbol{v}}$ other than $\vec{\mathbf{0}}$ is in the kernel). Prove that T has no inverse. *Hint: this means you can construct a vector that is not in the image of T!*

Problem 8. The previous problem will be very helpful for these two parts.

(a) Let $T\colon V \to V$ be an operator such that $\det[T] = 0$. Explain why there exists a solution to the homogeneous equation $S\vec{\boldsymbol{u}} = \vec{\mathbf{0}}$.

(b) Suppose $S\colon V \to V$ is another operator such that $\det[S] \neq 0$. Explain why there exists a solution to the inhomogeneous equation $S\vec{\boldsymbol{v}} = \vec{\boldsymbol{w}}$ for any $\vec{\boldsymbol{w}} \in V$.

Problem 9. Prove that the eigenvectors with eigenvalue 0 of an operator $T\colon V \to V$ correspond to vectors in the kernel of T.

Problem 10. Consider the linear operator $J\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$J(\hat{\boldsymbol{e}}_1) = \hat{\boldsymbol{e}}_2 \qquad \text{and} \qquad J(\hat{\boldsymbol{e}}_2) = -\hat{\boldsymbol{e}}_1.$$

(a) Show that operator polynomial

$$P(J) := J^2 + I\colon \mathbb{R}^2 \to \mathbb{R}^2$$

annihilates $\mathbb{R}^2$. Or, said another way, show that every $\vec{\boldsymbol{v}} \in \mathbb{R}^2$ is in the kernel of $P(J)$.

(b) Show that the characteristic polynomial of J is

$$p(\lambda) = \lambda^2 + 1.$$

Does this coincide with $P(J)$? *If need be, use your matrix representation $[J]$ from the previous homework.*

(c) Compute the eigenvalues λ_1 and λ_2 of J.

(d) Compute the corresponding eigenvectors of J.

(e) If we don't allow for complex scalars, J has no eigenvalues. However, J^2 does have only real eigenvalues. Using (a), show that J has eigenvalue $\lambda = -1$ with eigenvectors $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$.

(f) (Bonus) Can you argue that any nonzero rotation of $\mathbb{R}^2$ must have imaginary eigenvalues?

Problem 11. For this problem, we will consider eigenvectors of three operators that act on the space of analytic functions $C^\omega(\mathbb{C})$. Your goal should be to realize that these correspond to differential equations you have seen before.

(a) Take the operator $\frac{d}{dx}\colon C^\omega(\mathbb{C}) \to C^\omega(\mathbb{C})$. Show that the exponential $e^{kx} \in C^\omega(\mathbb{C})$ is an eigenvector (or eigenfunction) with eigenvalue k. Write down the corresponding ODE. *Hint: just by doing the problem properly, you will probably write down the ODE.*

(b) Take the operator $\frac{d^2}{dx^2}\colon C^\omega(\mathbb{C}) \to C^\omega(\mathbb{C})$. Show that there are two eigenfunctions $e^{i\omega x}$ and $e^{-i\omega x}$ with eigenvalue $-\omega^2$. Write down the corresponding ODE. *Hint: just by doing the problem properly, you will probably write down the ODE.*

(c) Take the operator $x\frac{d}{dx}\colon C^\omega(\mathbb{C}) \to C^\omega(\mathbb{C})$. Find the eigenfunctions to this operator using the fact that this corresponds to a separable ODE.

Problem 12. Consider the Legendre polynomials

$$B_L = \left\{ f_0 = \sqrt{\frac{1}{2}},\ f_1 = \sqrt{\frac{3}{2}}x,\ f_2 = \sqrt{\frac{5}{8}}(1-3x^2),\ f_3 = \sqrt{\frac{63}{8}}\left(x - \frac{5x^3}{3}\right) \right\}$$

which form a basis for $P_3(\mathbb{C})$.

(a) For polynomials $f, g \in P_3(\mathbb{C})$, define an inner product

$$\langle g, h\rangle \coloneqq \int_{-1}^{1} gh^* dx.$$

Show (or find in the text or previous homeworks) evidence that the basis B_L is orthonormal with respect to this inner product.

(b) Consider the operator

$$\mathcal{L} \coloneqq (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}\colon P_3(\mathbb{C}) \to P_3(\mathbb{C}).$$

Show that $\mathcal{L}$ is linear.

(c) Show that each Legendre polynomial f_i is an eigenvector (or eigenfunction) of $\mathcal{L}$. What are the corresponding eigenvalues? How do these eigenvalues correspond to the m that appears in Legendre's equation (see the section in our text).