MATH 271, Homework 8, *Solutions*

Problem 1. Consider the following vectors in space $\mathbb{R}^3$

$$\vec{\boldsymbol{u}} = \hat{\boldsymbol{e}}_1 + 2\hat{\boldsymbol{e}}_2 + 3\hat{\boldsymbol{e}}_3 \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -2\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 - 2\hat{\boldsymbol{e}}_3.$$

(a) Compute the dot product $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$.

(b) Compute the lengths $|\vec{\boldsymbol{u}}|$ and $|\vec{\boldsymbol{v}}|$ using the dot product.

(c) Compute the projection of $\vec{\boldsymbol{u}}$ in the direction of $\vec{\boldsymbol{v}}$. *Hint: don't forget to normalize the vectors before you build your projection.*

(d) Compute the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$.

(e) Find the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

Solution 1.

(a) We have that

$$\begin{aligned} \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} &= 1 \cdot (-2) + 2 \cdot 1 + 3 \cdot (-3) \\ &= -6. \end{aligned}$$

(b) We compute the lengths using the dot product by

$$\|\vec{\boldsymbol{u}}\| = \sqrt{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

Likewise

$$\|\vec{\boldsymbol{v}}\| = \sqrt{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}} = \sqrt{(-2)^2 + 1^2(-2)^2} = \sqrt{9} = 3.$$

(c) The projection of $\vec{\boldsymbol{u}}$ in the direction of $\vec{\boldsymbol{v}}$ is simply asking for how much of the vector $\vec{\boldsymbol{u}}$ is in the direction of $\vec{\boldsymbol{v}}$. One can arrive at this purely through trigonometry, but we have the dot product at our disposal. The normalized vector $\hat{\boldsymbol{v}}$ points in the direction of $\vec{\boldsymbol{v}}$ with length 1 and

$$\hat{\boldsymbol{v}} = \frac{1}{\|\vec{\boldsymbol{v}}\|}\vec{\boldsymbol{v}} = \frac{1}{3}\vec{\boldsymbol{v}}.$$

Then, the projection can be computed by

$$\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} = \frac{1}{3}\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = -2.$$

One should attempt to recover this notion by doing some trigonometry.

(d) Here, feel free to use a formula for a cross product instead of writing it all out. We will find that

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = -7\hat{\boldsymbol{x}} - 4\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}.$$

(e) The area of the parallelogram is given by the magnitude of the cross product so

$$A = |\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}| = 3\sqrt{10}.$$

Problem 2. Write down the matrix for the following linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}.$$

Solution 2. We need that

$$[T]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}$$

via matrix multiplication. Since the input vector is a 3-dimensional vector, and the output vector is 3-dimensional, we must have that $[T]$ is a 3×3-matrix. Hence,

$$[T] = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t_{11}x + t_{12}y + t_{13}z \\ t_{21}x + t_{22}y + t_{23}z \\ t_{31}x + t_{32}y + t_{33}z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}.$$

If we match the coefficients on the x, y, and z, we find that

$$[T] = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$

Problem 3. Consider the linear transformation $J\colon R^2 \to \mathbb{R}^2$ defined by

$$J(\hat{\boldsymbol{e}}_1) = \hat{\boldsymbol{e}}_2 \qquad \text{and} \qquad J(\hat{\boldsymbol{e}}_2) = -\hat{\boldsymbol{e}}_1.$$

This linear transformation is fundamental in understanding how we can reconstruct complex numbers using matrices.

(a) Show that $J^2 = J \circ J = -1$.

(b) Determine a matrix representation for J and denote it by $[J]$.

(c) Recall that we can represent a complex number as $z = x + iy$ and that we can represent z as a vector in $\mathbb{R}^2$ as $\vec{\boldsymbol{\zeta}} = x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2$. Show that $J\vec{\boldsymbol{\zeta}}$ corresponds to iz. *Hint: just show the multiplcations are analogous.*

(d) We can completely reconstruct a representation of $\mathbb{C}$ by using a matrix representation. In particular, we can take

$$[z] = x[I] + y[J].$$

Show that we recover the complex addition and multiplication using this representation.

(e) We can represent a unit complex number as $z = e^{i\theta}$. Show that the representation described before leads to

$$[z] = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Solution 3. (a) Let $\vec{\boldsymbol{v}} = v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}}$ be some arbitrary vector in $\mathbb{R}^2$. Then,

$$\begin{aligned} J^2(\vec{\boldsymbol{v}}) = J(J(\vec{\boldsymbol{v}})) &= J(J(v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}})) \\ &= J(v_1 J(\hat{\boldsymbol{x}}) + v_2 J(\hat{\boldsymbol{y}})) \\ &= J(v_1\hat{\boldsymbol{y}} - v_2\hat{\boldsymbol{x}}) \\ &= v_1 J(\hat{\boldsymbol{y}}) - v_2 J(\hat{\boldsymbol{x}}) \\ &= -v_1\hat{\boldsymbol{x}} - v_2\hat{\boldsymbol{y}} \\ &= -\vec{\boldsymbol{v}}. \end{aligned}$$

So, yes, J^2 acts like scaling by -1.

(b) We determine a matrix for J by using the definition of J on $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. In particular,

$$[J] = \begin{pmatrix} | & | \\ J(\hat{\boldsymbol{x}}) & J(\hat{\boldsymbol{y}}) \\ | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

One can check here that $[J]^2 = -[I]$, where $[I]$ is the identity matrix. This confirms that $[J]$ satisfies the relationship we saw in (a).

(c) In the complex plane, we let $z = x + iy$ and we can note that

$$iz = -y + ix.$$

Now, we can think of z as a vector in $\mathbb{R}^2$ by noticing that the vector $\vec{\boldsymbol{\zeta}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ corresponds to the same exact point geometrically. Then, if we apply J we have

$$J\vec{\boldsymbol{\zeta}} = -y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}},$$

which is exactly how z was transformed when we multiplied by i. Keep in mind that i rotates a complex number z by $\pi/2$ in the counterclockwise direction and J does the same to vectors $\vec{\boldsymbol{\zeta}}$. To see this most fully, consider drawing a picture of both transformations.

(d) In the complex plane, we can take two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we have

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \qquad \text{and} \qquad z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Notice that addition is componentwise and keep track of this result from the multiplication.

Now, we can consider two matrices $[z_1] = x_1[I] + y_1[J]$ and $[z_2] = x_2[I] + y_2[J]$ and see what we get through addition and multiplication. We have

$$[z_1] + [z_2] = (x_1 + x_2)[I] + (y_1 + y_2)[J].$$

This is due to how matrices add componentwise and we can see that this corresponds to the addition in $\mathbb{C}$. Next, we have

$$\begin{aligned}
[z_1][z_2] &= (x_1[I] + y_1[J])(x_2[I] + y_2[J]) \\
&= x_1 x_2[I]^2 + y_1 x_2[J][I] + x_1 y_2[I][J] + y_1 y_2[J]^2 \\
&= x_1 x_2[I] + y_1 x_2[J] + x_1 y_2[J] - y_1 y_2[I] \\
&= (x_1 x_2 - y_1 y_2)[I] + (x_1 y_2 + x_2 y_1)[J].
\end{aligned}$$

Note that I use the facts $[J][I] = [I][J] = [J]$, $[I]^2 = [I]$, and from (a) we know $[J]^2 = -[I]$. If we take a look at the end result, we can see that this is the same multiplication result as $z_1 z_2$ in $\mathbb{C}$.

(e) Using our knowledge from the previous problem, and Euler's formula, we know that we can take

$$[e^{i\theta}] = \cos(\theta)[I] + \sin(\theta)[J].$$

Writing out the matrices explicitly yields

$$[e^{i\theta}] = \begin{pmatrix} \cos(\theta) & 0 \\ 0 & \cos(\theta) \end{pmatrix} + \begin{pmatrix} 0 & -\sin(\theta) \\ \sin(\theta) & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

as intended.

Remark 1. If one goes to look up a rotation matrix for $\mathbb{R}^2$, you will find the matrix you found in (e). So, this goes to show that complex arithmetic captures rotations nicely through Euler's formula. Moreover, the matrix representation for a complex number is faithful in describing all that we need.

Problem 4. Take the following matrices:

$$[A] = \begin{pmatrix} 4 & 3 & 10 & 2 \\ 1 & 1 & 0 & 9 \end{pmatrix}, \quad [B] = \begin{pmatrix} 8 & 5 & 8 \\ 10 & 9 & 2 \\ 4 & 6 & 3 \end{pmatrix}, \quad [C] = \begin{pmatrix} 0 & 0 & 9 \\ 7 & 9 & 9 \\ 1 & 9 & 9 \\ 3 & 3 & 1 \end{pmatrix}$$

(a) Compute either $[A][C]$ or $[C][A]$ and state which multiplication is not possible.

(b) Compute either $[B][C]$ or $[C][B]$ and state which multiplication is not possible.

(c) Can you add any of these matrices?

(d) Describe each matrix as linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$. What is m and n for each? How does this relate to the number of rows and columns?

Solution 4.

(a) The matrix $[A]$ is a 2×4 matrix and matrix $[C]$ is a 4×3 matrix. So we can compute $[A][C]$ but not $[C][A]$. Given that, we also expect the output to be a 2×3 matrix. So, we have

$$[A][C] = \begin{pmatrix} 4 & 3 & 10 & 2 \\ 1 & 1 & 0 & 9 \end{pmatrix} \begin{pmatrix} 0 & 0 & 9 \\ 7 & 9 & 9 \\ 1 & 9 & 9 \\ 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 37 & 123 & 155 \\ 34 & 36 & 27 \end{pmatrix}.$$

(b) $[B]$ is a 3×3 matrix so we can take $[C][B]$ but not $[B][C]$. We get

$$[C][B] = \begin{pmatrix} 0 & 0 & 9 \\ 7 & 9 & 9 \\ 1 & 9 & 9 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 8 & 5 & 8 \\ 10 & 9 & 2 \\ 4 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 36 & 54 & 27 \\ 182 & 170 & 101 \\ 134 & 140 & 53 \\ 58 & 48 & 33 \end{pmatrix}.$$

(c) We can always add a matrix to itself, so, for example $[A]+[A]$, $[B]+[B]$, and $[C]+[C]$ make sense. However, since the dimensions of $[A]$, $[B]$, and $[C]$ all differ, we cannot add in any other way.

(d) The number of columns of a matrix denotes the input dimension m, and the number of rows denotes the output dimension n. So

$$A\colon \mathbb{R}^4 \to \mathbb{R}^2, \quad B\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad C\colon \mathbb{R}^3 \to \mathbb{R}^4.$$

Problem 5. Solve the following equation.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 11 \end{pmatrix}.$$

Solution 5. First, we create the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 1 & 8 \\ 1 & 2 & 2 & 11 \end{array}\right).$$

We can subtract R1 from both R2 and R3 to get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \end{array}\right).$$

Then subtract R3 from R1 to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \end{array}\right).$$

Finally, subtract R2 from R3 to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

This yields our result in the right most column in that $x = 1$, $y = 2$, and $z = 3$.

Problem 6. Consider the space of polynomials of degree at most 3, $P_3(\mathbb{C})$.

(a) Using the basis

$$B = \{1, x, x^2, x^3\},$$

determine a matrix representation for the linear transformation $\frac{d}{dx}\colon P_3(\mathbb{C}) \to P_3(\mathbb{C})$.

(b) Show that the set of Legendre polynomials

$$B_L = \left\{ f_0 = \sqrt{\frac{1}{2}},\ f_1 = \sqrt{\frac{3}{2}}x,\ f_2 = \sqrt{\frac{5}{8}}(1-3x^2),\ f_3 = \sqrt{\frac{63}{8}}\left(x - \frac{5x^3}{3}\right) \right\}$$

is a basis for $P_3(\mathbb{C})$.

(c) Using the basis B_L instead, compute a matrix representation for the linear transformation $\frac{d}{dx}$.

Solution 6.

(a) Using the basis B, we can take an arbitrary degree at most 3 polynomial and write it as a column vector by

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

that is, we let

$$1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad x^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad x^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then we can compute

$$\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2 = \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{pmatrix}.$$

To get a matrix for this expression, we have

$$\left[\frac{d}{dx}\right] \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{pmatrix}$$

which one can verify is

$$\left[\frac{d}{dx}\right] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) Let $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be an arbitrary polynomial. To show B_L is a basis, we need to show that using elements of B_L we can generate this arbitrary polynomial by a linear combination in a unique way. So we must solve

$$b_0 f_0 + b_1 f_1 + b_2 f_2 + b_3 f_3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

for the b_j terms. Writing this out more completely,

$$\begin{aligned} b_0 f_0 + b_1 f_1 + b_2 f_2 + b_3 f_3 &= b_0 \sqrt{\frac{1}{2}} + b_1 \sqrt{\frac{3}{2}} x + b_2 \sqrt{\frac{5}{8}} (1 - 3x^2) + b_3 \sqrt{\frac{63}{8}} \left(x - \frac{5x^3}{3} \right) \\ &= \left(b_0 \sqrt{\frac{1}{2}} + b_2 \sqrt{\frac{5}{8}} \right) + \left(b_1 \sqrt{\frac{3}{2}} + b_3 \sqrt{\frac{63}{8}} \right) x + b_2 3 \sqrt{\frac{5}{8}} x^2 + b_3 \frac{5}{3} \sqrt{\frac{63}{8}} x^3. \end{aligned}$$

Now, looking at the x^2 and x^3 terms we can see that

$$b_2 = \frac{1}{3} \sqrt{\frac{8}{5}} a_2 \qquad \text{and} \qquad b_3 = \frac{3}{5} \sqrt{\frac{8}{63}} a_3.$$

It then follows that

$$b_0 = \sqrt{2} \left(a_0 - \frac{1}{3} a_2 \right) \qquad \text{and} \qquad b_1 = \sqrt{\frac{2}{3}} \left(a_1 - \frac{3}{5} a_3 \right).$$

This shows we have found b_j uniquely so B_L is indeed a basis.

(c) Now, if we take this new basis B_L then we have

$$b_0 f_0 + b_1 f_1 + b_2 f_2 + b_3 f_3 = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where again

$$f_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

This time, let us compute the transformation of each individual basis element

$$\begin{aligned} \frac{d}{dx} f_0 &= 0 \\ \frac{d}{dx} f_1 &= \sqrt{\frac{3}{2}} \\ \frac{d}{dx} f_2 &= -6 \sqrt{\frac{5}{8}} x \\ \frac{d}{dx} f_3 &= \sqrt{\frac{63}{8}} \left(1 - 5x^2 \right). \end{aligned}$$

Then our goal is to write each one of these answers in terms of the basis B_L as well. That is,

$$\begin{aligned}
\frac{d}{dx}f_0 &= 0 = 0f_0 + 0f_1 + 0f_2 + 0f_3 \\
\frac{d}{dx}f_1 &= \sqrt{\frac{3}{2}} = \sqrt{\frac{1}{3}}f_0 + 0f_1 + 0f_2 + 0f_3 \\
\frac{d}{dx}f_2 &= -6\sqrt{\frac{5}{8}}x = 0f_0 + \sqrt{15}f_1 + 0f_2 + 0f_3 \\
\frac{d}{dx}f_3 &= \sqrt{\frac{63}{8}}\left(1 - 5x^2\right) = \frac{1}{2}\sqrt{14}f_0 + 0f_1 + \sqrt{35}f_2 + 0f_3.
\end{aligned}$$

Hence, we use these as columns for the matrix

$$\left[\frac{d}{dx}\right]_{B_L} = \begin{pmatrix} 0 & \sqrt{\frac{1}{3}} & 0 & \frac{1}{2}\sqrt{14} \\ 0 & 0 & \sqrt{15} & 0 \\ 0 & 0 & 0 & \sqrt{35} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 7. Let $C^\omega(\mathbb{C})$ be the set of analytic functions (functions that have a power series representation), i.e., functions of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n \in \mathbb{C}$ for $n = 0, 1, 2, \ldots$. Let us compare this with the vector space of polynomials $P_N(\mathbb{C})$.

(a) Argue that $C^\omega(\mathbb{C})$ is a vector space. *Hint: show what addition and scalar multiplication look like.*

(b) Show that the space of polynomials of degree at most N, $P_N(\mathbb{C})$ is a subspace of $C^\omega(\mathbb{C})$.

(c) Let $f, g \in C^\omega(\mathbb{C})$ be given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n,$$

then define an inner product on $C^\omega(\mathbb{C})$ by taking

$$\langle f, g \rangle \coloneqq \sum_{n=0}^{\infty} a_n b_n^*.$$

Now, write $h(x)$ as a Taylor series centered at $x = 0$ and show that

$$h^{(k)}(0) = k!\langle h, x^k \rangle.$$

(d) Show that the N^{th} order Taylor approximation for the function $h(x)$ centered at $x = 0$ is the projection onto the subspace spanned by the functions

$$S = \{1, x, x^2, \ldots, x^N\}.$$

This projection is given by

$$\text{proj}_S(h) = \sum_{n=0}^{N} \langle h, x^n \rangle x^n.$$

Solution 7.

(a) For $C^\omega(\mathbb{C})$ to be a vector space, we need the properties of addition and scalar multiplication to hold and we need to identify the zero vector and identity element of the field. First, note that $0 \in C^\omega(\mathbb{C})$ and let $f, g \in C^\omega(\mathbb{C})$ be given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n.$$

It is clear that $0 + f = f$ so the function 0 acts as the zero vector. Similarly, $1 \in \mathbb{C}$ satisfies $1f = f$. Then we can note that addition of functions is associative and

$$(\alpha + \beta) f = (\alpha + \beta) \sum_{n=0}^{\infty} a_n x^n = \alpha \sum_{n=0}^{\infty} a_n x^n + \beta \sum_{n=0}^{\infty} a_n x^n = \alpha f + \beta f$$

likewise

$$\alpha(f + g) = \alpha \left(\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n \right) = \alpha \sum_{n=0}^{\infty} a_n x^n + \alpha \sum_{n=0}^{\infty} b_n x^n.$$

In fact, the key insight is that a linear combination

$$\alpha f + \beta g = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) x^n$$

so we are merely adding linear combinations of the sequences a_n and b_n together. If you'd like, you could use the sequence where 1 is in the jth entry of a sequence as a basis and write

$$f = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} \qquad \text{and} \qquad g = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix}.$$

In this sense, the elements in $C^\omega(\mathbb{C})$ are nothing but sequences or infinitely tall column vectors. However, they are special sequences that converge quickly enough to zero!

(b) We have already argued that $P_N(\mathbb{C})$ is a vector space itself and to see that it is a subspace, just note that any polynomial function is automatically a power series. A polynomial of degree at most N is just a power series where all the coefficients where $k > N$ must $a_k = 0$. That is

$$p(x) = \sum_{n=0}^{N} c_n x^n$$

is a polynomial of degree at most N and if we take $c_k = 0$ for $k > N$, then

$$\sum_{n=0}^{\infty} c_n x^n$$

shows that the polynomials are a subspace. Another way to see this is,

$$p = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_N \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

(c) Based on the previous part, the function x^k is given by a power series

$$x^k = \sum_{n=0}^{\infty} \alpha_n x^n$$

where $\alpha_k = 1$ but $\alpha_j = 0$ for all $j \neq k$. Then, for $h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} x^n$ we have

$$\langle h, x^k \rangle = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} \alpha_n^*.$$

But the only term in the series that survives is when $n = k$ since $\alpha_j = 0$ for all $j \neq k$ therefore

$$\langle h, x^k \rangle = \frac{h^{(k)}(0)}{n!} \alpha_n$$

Hence,

$$h^{(k)}(0) = k! \langle h, x^k \rangle.$$

(d) Using the logic from the previous part,

$$\mathrm{proj}_S(h) = \sum_{n=0}^{N} \langle h, x^n \rangle x^n = \sum_{n=0}^{N} \frac{h^{(k)}(0)}{n!} x^n.$$