MATH 271, Homework 8

Due November 16th

Problem 1. Consider the following vectors in space $\mathbb{R}^3$

$$\vec{\boldsymbol{u}} = \hat{\boldsymbol{e}}_1 + 2\hat{\boldsymbol{e}}_2 + 3\hat{\boldsymbol{e}}_3 \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -2\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 - 2\hat{\boldsymbol{e}}_3.$$

(a) Compute the dot product $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$.

(b) Compute the lengths $|\vec{\boldsymbol{u}}|$ and $|\vec{\boldsymbol{v}}|$ using the dot product.

(c) Compute the projection of $\vec{\boldsymbol{u}}$ in the direction of $\vec{\boldsymbol{v}}$. *Hint: don't forget to normalize the vectors before you build your projection.*

(d) Compute the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$.

(e) Find the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

Problem 2. Write down the matrix for the following linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}.$$

Problem 3. Consider the linear transformation $J\colon R^2 \to \mathbb{R}^2$ defined by

$$J(\hat{\boldsymbol{e}}_1) = \hat{\boldsymbol{e}}_2 \qquad \text{and} \qquad J(\hat{\boldsymbol{e}}_2) = -\hat{\boldsymbol{e}}_1.$$

This linear transformation is fundamental in understanding how we can reconstruct complex numbers using matrices.

(a) Show that $J^2 = J \circ J = -1$.

(b) Determine a matrix representation for J and denote it by $[J]$.

(c) Recall that we can represent a complex number as $z = x + iy$ and that we can represent z as a vector in $\mathbb{R}^2$ as $\vec{\boldsymbol{\zeta}} = x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2$. Show that $J\vec{\boldsymbol{\zeta}}$ corresponds to iz. *Hint: just show the multiplcations are analogous.*

(d) We can completely reconstruct a representation of $\mathbb{C}$ by using a matrix representation. In particular, we can take

$$[z] = x[I] + y[J].$$

Show that we recover the complex addition and multiplication using this representation.

(e) We can represent a unit complex number as $z = e^{i\theta}$. Show that the representation described before leads to

$$[z] = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Problem 4. Take the following matrices:

$$[A] = \begin{pmatrix} 4 & 3 & 10 & 2 \\ 1 & 1 & 0 & 9 \end{pmatrix}, \quad [B] = \begin{pmatrix} 8 & 5 & 8 \\ 10 & 9 & 2 \\ 4 & 6 & 3 \end{pmatrix}, \quad [C] = \begin{pmatrix} 0 & 0 & 9 \\ 7 & 9 & 9 \\ 1 & 9 & 9 \\ 3 & 3 & 1 \end{pmatrix}$$

(a) Compute either $[A][C]$ or $[C][A]$ and state which multiplication is not possible.

(b) Compute either $[B][C]$ or $[C][B]$ and state which multiplication is not possible.

(c) Can you add any of these matrices?

(d) Describe each matrix as linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$. What is m and n for each? How does this relate to the number of rows and columns?

Problem 5. Solve the following equation.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 11 \end{pmatrix}.$$

Problem 6. Consider the space of polynomials of degree at most 3, $P_3(\mathbb{C})$.

(a) Using the basis

$$B = \{1, x, x^2, x^3\},$$

determine a matrix representation for the linear transformation $\frac{d}{dx}\colon P_3(\mathbb{C}) \to P_3(\mathbb{C})$.

(b) Show that the set of Legendre polynomials

$$B_L = \left\{ f_0 = \sqrt{\frac{1}{2}},\ f_1 = \sqrt{\frac{3}{2}}x,\ f_2 = \sqrt{\frac{5}{8}}(1 - 3x^2),\ f_3 = \sqrt{\frac{63}{8}}\left(x - \frac{5x^3}{3}\right) \right\}$$

is a basis for $P_3(\mathbb{C})$.

(c) Using the basis B_L instead, compute a matrix representation for the linear transformation $\frac{d}{dx}$.

Remark 1. This should go to show you that a matrix representation depends on a basis!!!

Problem 7. Let $C^\omega(\mathbb{C})$ be the set of analytic functions (functions that have a power series representation), i.e., functions of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n \in \mathbb{C}$ for $n = 0, 1, 2, \ldots$. Let us compare this with the vector space of polynomials $P_N(\mathbb{C})$.

(a) Argue that $C^\omega(\mathbb{C})$ is a vector space. *Hint: show what addition and scalar multiplication look like.*

(b) Show that the space of polynomials of degree at most N, $P_N(\mathbb{C})$ is a subspace of $C^\omega(\mathbb{C})$.

(c) Let $f, g \in C^\omega(\mathbb{C})$ be given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n,$$

then define an inner product on $C^\omega(\mathbb{C})$ by taking

$$\langle f, g \rangle \coloneqq \sum_{n=0}^{\infty} a_n b_n^*.$$

Now, write $h(x)$ as a Taylor series centered at $x = 0$ and show that

$$h^{(n)}(0) = n! \langle h, x^n \rangle.$$

(d) Show that the N^{th} order Taylor approximation for the function $h(x)$ centered at $x = 0$ is the projection onto the subspace spanned by the functions

$$S = \{1, x, x^2, \ldots, x^N\}.$$

This projection is given by

$$\operatorname{proj}_S(h) = \sum_{n=0}^{N} \langle h, x^n \rangle x^n.$$