MATH 271, Homework 7

Due November 3rd

Problem 1. Consider the following vectors in the real plane $\mathbb{R}^2$. We let

$$\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -3\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}}.$$

(a) What is the dimension of the vector space $\mathbb{R}^2$? Explain.

(b) Draw both $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ in the plane and label the origin.

(c) Draw the vector $\vec{\boldsymbol{w}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$ in the plane.

(d) Draw the subspace spanned by $\vec{\boldsymbol{u}}$.

Problem 2. Let a mass m_1 weighing $1kg.$ be placed at $\vec{\boldsymbol{r}}_1 = 2\hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$ and a mass m_2 of $2kg.$ be placed at $\vec{\boldsymbol{r}}_2 = 4\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}$. Where must a mass m_3 of $3kg.$ be placed so that the center of mass is at the origin $\vec{\boldsymbol{0}}$?

Problem 3. Which of the following are linear transformations? For those that are not, which properties of linearity (the properties (i) and (ii) in our notes) fail? Show your work.

(a) $T_a \colon \mathbb{R} \to \mathbb{R}$ given by $T_a(x) = \frac{1}{x}$.

(b) $T_b \colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T_b \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(c) $T_c \colon \mathbb{R} \to \mathbb{R}^3$ given by

$$T_c(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}.$$

(d) $T_d \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T_d \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix}.$$

Problem 4.

(a) We can reflect an arbitrary vector in the plane by defining a function that reflects the basis vectors and extending the function with linearity. Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function be defined by

$$R(\hat{\boldsymbol{x}}) = -\hat{\boldsymbol{x}} \qquad \text{and} \qquad R(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{y}}.$$

Let $\vec{\boldsymbol{v}} = \alpha_1 \hat{\boldsymbol{x}} + \alpha_2 \hat{\boldsymbol{y}}$ and let

$$R(\vec{\boldsymbol{v}}) = \alpha_1 R(\hat{\boldsymbol{x}}) + \alpha_2 R(\hat{\boldsymbol{y}}).$$

Show that R reflects the vector $\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}}$ about the y-axis and draw a picture.

(b) We can rotate a vector in the plane by first rotating the basis vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Define a linear function $J\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$J(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}} \qquad \text{and} \qquad J(\hat{\boldsymbol{y}}) = -\hat{\boldsymbol{x}}.$$

Show that J rotates $\vec{\boldsymbol{u}}$ by $\pi/2$ in the counterclockwise direction and draw a picture.

Problem 5. Let S be the set of general solutions to the following second order homogeneous linear differential equation

$$x'' + f(t)x' + g(t)x = 0.$$

Show that this set S is a vector space over the field of complex numbers.

Problem 6. Let $P_3(\mathbb{C})$ be the vector space of polynomials of degree at most 3 with coefficients in $\mathbb{C}$ with variable x. For example,

$$f(x) = x^2 + 1 \in P_3(\mathbb{C}).$$

(a) Write down a basis for $P_3(\mathbb{C})$.

(b) What is the dimension of the vector space $P_3(\mathbb{C})$.

(c) Let $\frac{d}{dx}\colon P_3(\mathbb{C}) \to P_3(\mathbb{C})$. Argue that $\frac{d}{dx}$ is a linear transformation.

(d) What is the kernel (nullspace) of $\frac{d}{dx}$? What is the image (range) of $\frac{d}{dx}$?