MATH 271, HOMEWORK 5, *Solutions*
DUE OCTOBER 11TH

Problem 1 (Euler's Formula)**.** Given that

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{and} \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

(a) Plot the approximations of both $\cos(x)$ and $\sin(x)$ versus the original function for order 1,5,20 over the domain $[-4\pi, 4\pi]$.

(b) Show that
$$e^{ix} = \cos(x) + i\sin(x),$$
using the power series representation for the exponential function e^x.

(c) Show that cosine is even, $\cos(-x) = \cos(x)$, and that sine is odd $\sin(-x) = -\sin(x)$.

(d) Compute $\frac{d}{dx}e^{ix}$ using the series representation and show that $\frac{d}{dx}\cos(x) = -\sin(x)$ and $\frac{d}{dx}\sin(x) = \cos(x)$.

Solution 1.

(a) Here is the plot of the approximations of $\sin(x)$. *Note that order refers to the highest power of x seen in the approximations* <u>*NOT*</u> *the N used for a partial sum!*

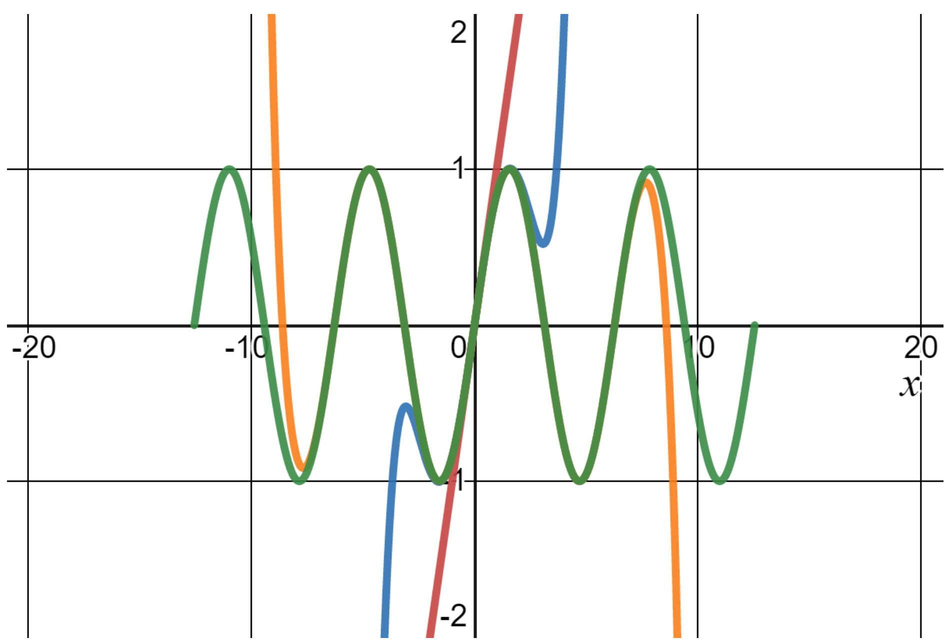


Figure 1: 1st order in red, 5th order in blue, and 20th order in orange compared to $\sin(x)$ itself in green.

And now the figure for the approximations of $\cos(x)$.

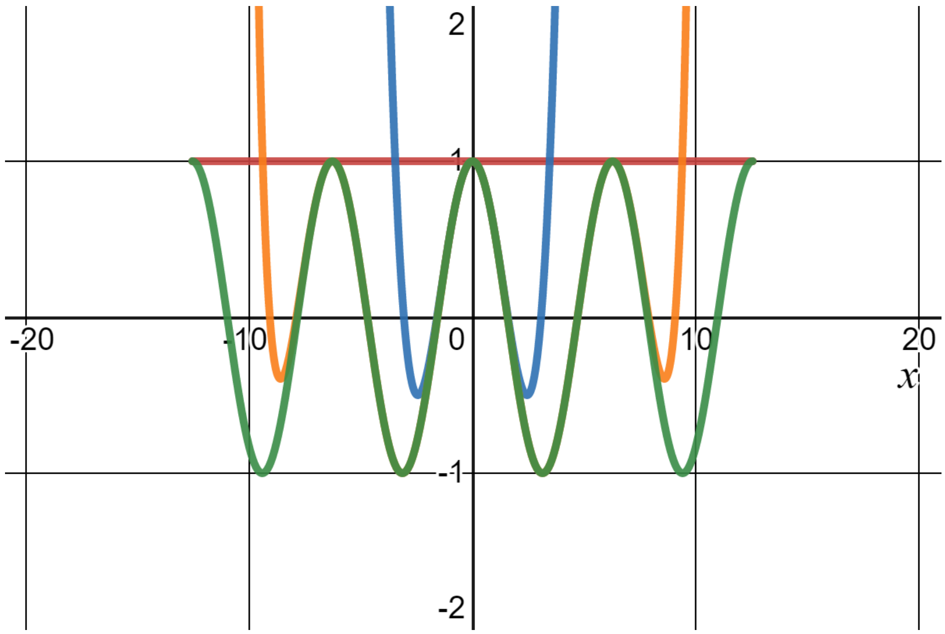


Figure 2: 1^{st} order in red, 5^{th} order in blue, and 20^{th} order in orange compared to $\cos(x)$ itself in green.

(b)

(c) We take

$$\begin{aligned}\cos(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n(-x)^{2n}}{(2n)!} &= \sum_{n=0}^{\infty} \frac{(-1)^n\left((-x)^2\right)^n}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ &= \cos(x).\end{aligned}$$

Fundamentally, this is because all powers of x in the terms in the series are even. This is why we call cos an *even* function!

(d) Similarly, we take

$$\begin{aligned}\sin(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n(-x)^{2n+1}}{(2n+1)!} &= \sum_{n=0}^{\infty} \frac{(-1)^n(-x)\cdot(-x)^{2n}}{(2n+1)!} \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n x\cdot x^{2n}}{(2n+1)!} \\ &= -\sum_{n=0}^{\infty} \frac{(-1)x^{2n+1}}{(2n+1)!} \\ &= -\sin(x).\end{aligned}$$

Again, this is happening due to the fact that all powers of x in the terms for the series are odd. This is why we call sin an *odd* function.

(e) Consider first $\frac{d}{dx}\sin(x)$. We take

$$\begin{aligned}\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\\ &= \frac{d}{dx}x - \frac{d}{dx}\frac{x^3}{3!} + \frac{d}{dx}\frac{x^5}{5!} - \cdots\\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots .\end{aligned}$$

These are the first few terms of the $\cos(x)$ series. Notice if we take

$$\frac{d}{dx}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \frac{(-1)x^{2n}}{(2n)!}.$$

So we have

$$\frac{d}{dx}\sin(x) = \cos(x).$$

When we consider $\frac{d}{dx}$ of $\cos(x)$ we have to be a bit more careful. Let's see what happens. We take

$$\begin{aligned}\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} &= \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)\\ &= \frac{d}{dx}1 - \frac{d}{dx}\frac{x^2}{2!} + \frac{d}{dx}\frac{x^4}{4!} - \cdots\\ &= 0 - x + \frac{x^3}{3!} - \cdots\end{aligned}$$

which look like the first terms in the series for $-\sin(x)$. However, let's take a derivative of the term

$$\frac{d}{dx}\frac{(-1)^n x^{2n}}{(2n)!} = \frac{(-1)^n x^{2n-1}}{(2n-1)!}.$$

Now if we were to have this in our series we find

$$\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \neq \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!},$$

since the first term on the right has an x^{-1} in it! When we write out the terms of the series and differentiate them, we don't make this mistake. We just have to be careful. What we really should have is

$$\begin{aligned}\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} &= \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}\\ &= -\sin(x).\end{aligned}$$

Notice above I *reindexed* the series. This is not something I'm going to worry too much about you doing.

Remark 1. Again, some of my work here is beyond what I was expecting. If you would have taken the derivatives of the first few terms and showed they are equal, you can extrapolate beyond that and assume it will work for the rest of the terms. Just be careful with this!

Problem 2. Consider the function

$$f(x) = \frac{1}{1-x}.$$

(a) Compute the Maclaurin series for the function.

(b) Find the integral $\int \frac{dx}{1-x}$ using the Maclaurin series for $f(x)$ found in (a).

(c) Write down the Maclaurin series for $\ln(1-x)$ and compare to your answer in (b).

Solution 2.

(a) To find the Maclaurin series (i.e., the Taylor series centered at $a = 0$), we must compute $f^{(n)}(0)$ as we desire to find

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The derivatives are

$$\begin{aligned}
f^{(0)}(x) &= \frac{1}{1-x} && \Longrightarrow && f^{(0)}(0) = 1 = 0! \\
f^{(1)}(x) &= \frac{1}{(1-x)^2} && \Longrightarrow && f^{(1)}(0) = 1 = 1! \\
f^{(2)}(x) &= \frac{2}{(1-x)^3} && \Longrightarrow && f^{(2)}(0) = 2 = 2! \\
f^{(3)}(x) &= \frac{6}{(1-x)^4} && \Longrightarrow && f^{(3)}(0) = 6 = 3! \\
f^{(4)}(x) &= \frac{24}{(1-x)^5} && \Longrightarrow && f^{(4)}(0) = 24 = 4! \\
& && \vdots && \\
f^{(n)}(x) &= \frac{n!}{(1-x)^{n+1}} && \Longrightarrow && f^{(n)}(0) = n!.
\end{aligned}$$

Hence, if we plug this into the formula for the Maclaurin series we have

$$f(x) = \sum_{n=0}^{\infty} x^n.$$

(b) We can integrate this series term by term to find the desired antiderivative. So we have

$$\int \frac{dx}{1-x} = \int \sum_{n=0}^{\infty} x^n dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

(c) We can find the Maclaurin series for $\ln(1-x)$ in the same way as above. However, notice that $\frac{d}{dx} \ln(1-x) = \frac{-1}{1-x}$, and thus up to determining the constant C, we have that

$$-\left(C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\right) = \ln(1-x).$$

Problem 3.

(a) Compute the Taylor series centered at $a = 0$ for $f(x) = e^{-\frac{x^2}{2}}$.

(b) Use the Taylor series for e^x and modify it to find a power series for $f(x)$. Is this the same as the series in (a)?

(c) Plot the original function $f(x)$ compared to the first, second, third, and fourth term approximation for the series on the same graph.

Solution 3. (a) We find the Taylor series centered at $a = 0$ by computing $f^{(n)}(0)$. The derivatives are

$$\begin{aligned}
f^{(0)}(x) &= e^{-\frac{x^2}{2}} & &\Longrightarrow & f^{(0)}(0) &= 1 \\
f^{(1)}(x) &= xe^{-\frac{x^2}{2}} & &\Longrightarrow & f^{(1)}(0) &= 0 \\
f^{(2)}(x) &= (x^2-1)e^{-\frac{x^2}{2}} & &\Longrightarrow & f^{(2)}(0) &= -1 \\
f^{(3)}(x) &= x(x^2-3)e^{-\frac{x^2}{2}} & &\Longrightarrow & f^{(3)}(0) &= 0 \\
f^{(4)}(x) &= (x^4-6x^2+3)e^{-\frac{x^2}{2}} & &\Longrightarrow & f^{(4)}(0) &= 3.
\end{aligned}$$

This gives us the first five terms of the Taylor series for $f(x)$ so that we have

$$f(x) \approx 1 + 0 + \frac{-1}{2}x^2 + 0 + \frac{3}{4!}x^4.$$

(b) The easier way is to modify an already known power series like e^x. We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

If we replace $x \mapsto -\frac{x^2}{2}$ then we have

$$e^{-\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}.$$

(c) Below is a graph showing the three different functions.

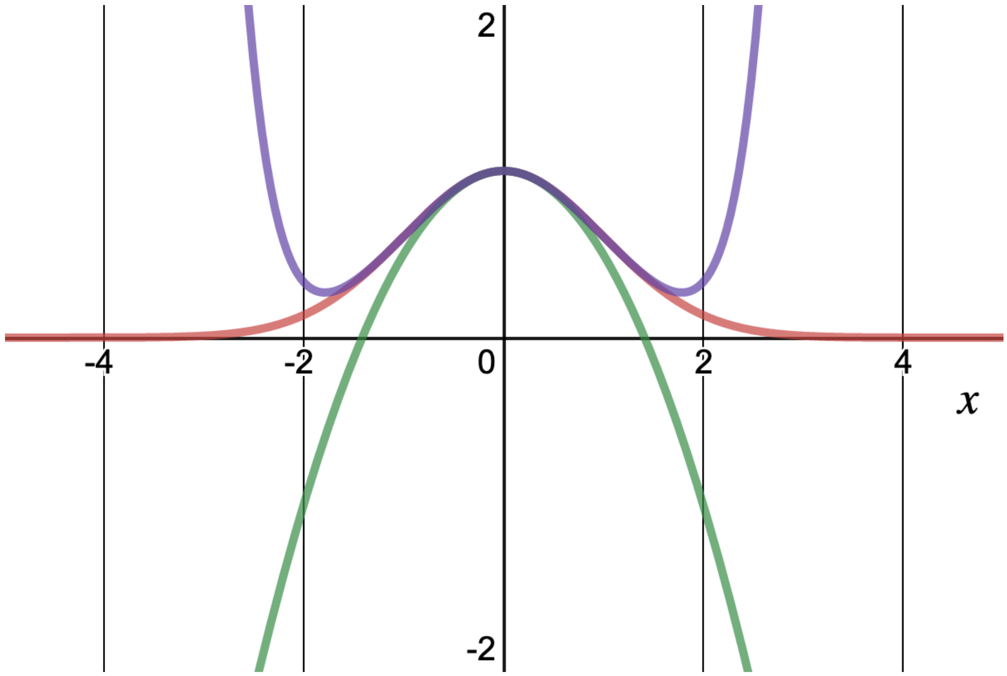


Figure 3: Red: $f(x)$; Green: Four terms from Taylor series; Purple: Four terms from modified Taylor series.

Remark 2. The reason why the graphs are different is because the modified series skips the zero terms that show up in the Taylor series computation. So when we plot four terms in the modified series, it is equal to plotting the first eight terms of the actual Taylor series.

Problem 4. How can we approximate a (possibly complicated) function by using a power series? Why is this useful (specifically for computation on a computer)?

Solution 4. We can often times approximate a function about a point $x = a$ using a truncated Taylor series centered about a. What I mean is that we will have a function $f(x)$ which (inside its interval of convergence) will be equal to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This proves to be very useful as we can take finite truncations of the complete power series above. Specifically, we have that

$$f(x) \approx \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

It turns out that this is a reasonable approximation for f as long as we don't look too far away from $x = a$. Also, computers really only have the ability to add. Of course, multiplication is sequential adding, division can be done through subtraction and multiplication, and powers come from sequential multiplication. The point is, computers work with polynomials (or rational functions) and this gives us a way to realize a complicated non-polynomial function as (approximately) a polynomial function.

Problem 5 (Explicit Euler Method)**.** Consider the differential equation $x' = kx$ where $k \in \mathbb{C}$ is a complex parameter for the system. Note that the solution to this equation given the initial condition $x(0) = 1$ is $x = e^{kt}$.

(a) Suppose we want to find an approximation to a solution using a computer. Let t_0 be some arbitrary time, define δt to some fixed change in the input variable t and let $\delta x = x(t_0 + \delta t) - x(t_0)$ be the corresponding change in the output x. Compute the first order Taylor approximation of x at the point t_0 to see that

$$x(t_0 + \delta t) \approx x(t_0) + x'(t_0)(\delta t) \tag{1}$$

from which you can then note that

$$x'(t_0) \approx \frac{\delta x}{\delta t} \tag{2}$$

(b) Define the *explicit Euler approximation sequence* $\{x_\tau\}_{\tau=0}^{T}$ so that $x(t_0) = x_0$ and at later times $x(t_0 + \tau\delta t) = x_\tau$. Show using the previous equations, the ODE itself, and the fact that $x_{\tau-1} + \delta x = x_\tau$ means can make a sequence

$$x_\tau = x_{\tau-1} + kx_{\tau-1}\delta t.$$

(c) Let $k = 1$, $\delta t = 0.01$, $t_0 = 0$, and let $x(0) = 1$. Plot the explicit Euler approximation sequence using the following URL `http://www.calcul.com/show/calculator/recursive`. Compare this graph to the solution $x(t) = e^t$.

(d) Let $k = -1$, $\delta t = 2$, $t_0 = 0$, and let $x(0) = 1$. Plot the approximation again. Is there something wrong? How does this compare to what the solution should be?

Solution 5.

(a) The Taylor series for the function x centered at t_0 is given by

$$x(t) = \sum_{n=0}^{\infty} \frac{x'(t_0)}{n!}(t - t_0)^{(n)}.$$

Hence, to first order

$$x(t) \approx x(t_0) + x'(t_0)(t - t_0).$$

Now, let $t = t_0 + \delta t$ and we have

$$\boxed{x(t_0 + \delta t) \approx x(t_0) + x'(t_0)\delta t.}$$

(b) Note that

$$x_\tau = x_{\tau-1} + \delta x$$

and by eq. (2) that in general

$$\delta x = x'(t)\delta t.$$

Hence, by the previous two equations

$$x_\tau = x_{\tau-1} + x'(t_0 + (\tau - 1)\delta t)\delta t$$

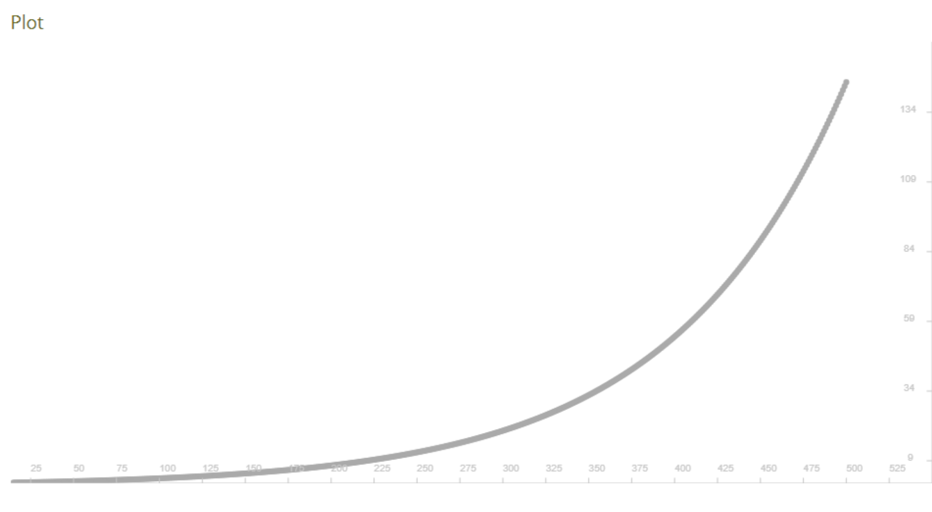
. By the ODE itself $x' = kx$ therefore $x'(t_0 + (\tau - 1)\delta t) = kx_{\tau-1}$ and we have our intended result

$$\boxed{x_\tau = x_{\tau-1} + kx_{\tau-1}\delta t.}$$

(c) Take our parameters as given so our equation assumes the form

$$x_\tau = x_{\tau-1} + 0.01 \cdot x_{\tau-1} = 1.01x_{\tau-1}$$

along with the initial condition $x_0 = 1$. Notice that we have turned exponential growth into a sequence of "geometric" growth. Here is a graph of 500 points of the approximation.



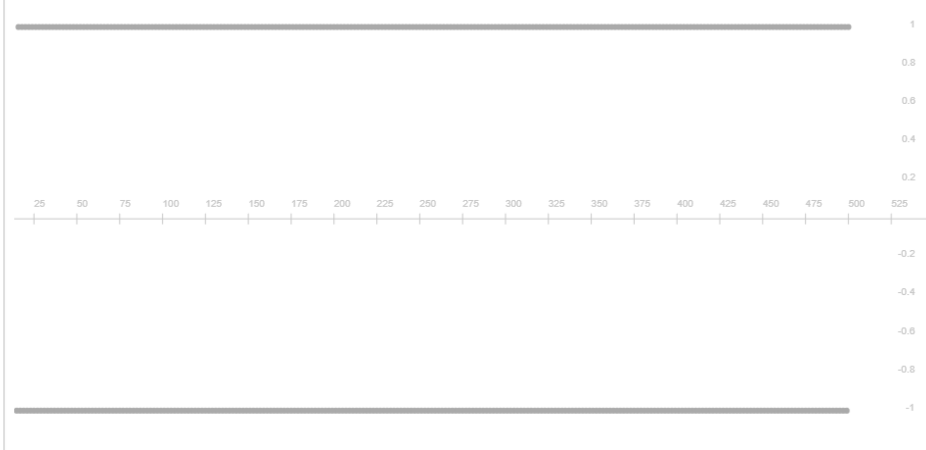
The graph here matches the true solution $x(t)e^t$ very closely.

(d) Let us now take the next set of parameters so that

$$x_\tau = x_{\tau-1} - 2x_{\tau-1} = -x_{\tau-1}$$

along with the initial condition $x_0 = 1$. Now, we can notice that this sequence will behave quite differently since $x_\tau = -x_{\tau-1}$ shows that our sequence will just oscillate between two numbers.

Plot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