MATH 271, HOMEWORK 5

DUE OCTOBER 11^{TH}

Problem 1 (Euler's Formula)**.** Given that

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{and} \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

(a) Plot the approximations of both $\cos(x)$ and $\sin(x)$ versus the original function for order 1,5,20 over the domain $[-4\pi, 4\pi]$.

(b) Show that

$$e^{ix} = \cos(x) + i\sin(x),$$

using the power series representation for the exponential function e^x.

(c) Show that cosine is even, $\cos(-x) = \cos(x)$, and that sine is odd $\sin(-x) = -\sin(x)$.

(d) Compute $\frac{d}{dx}e^{ix}$ using the series representation and show that $\frac{d}{dx}\cos(x) = -\sin(x)$ and $\frac{d}{dx}\sin(x) = \cos(x)$.

Problem 2. Consider the function

$$f(x) = \frac{1}{1-x}.$$

(a) Compute the Taylor series centered at $a = 0$ for the function.

(b) Find the antiderivative $\int \frac{dx}{1-x}$ using the Taylor series for $f(x)$ found in (a).

(c) Write down the Taylor series for $\ln(1-x)$ centered at $a = 0$ and compare to your answer in (b).

Problem 3.

(a) Compute the Taylor series centered at $a = 0$ for $f(x) = e^{-\frac{x^2}{2}}$.

(b) Use the Taylor series for e^x and modify it to find a power series for $f(x)$. Is this the same as the series in (a)?

(c) Plot the original function $f(x)$ compared to the first, second, third, and fourth term approximation for the series on the same graph.

Problem 4. How can we approximate a (possibly complicated) function by using a power series? Why is this useful (specifically for computation on a computer)?

Problem 5 (Explicit Euler Method)**.** Consider the differential equation $x' = kx$ where $k \in \mathbb{C}$ is a complex parameter for the system. Note that the solution to this equation given the initial condition $x(0) = 1$ is $x = e^{kt}$.

(a) Suppose we want to find an approximation to a solution using a computer. Let t_0 be some arbitrary time, define δt to some fixed change in the input variable t and let $\delta x = x(t_0 + \delta t) - x(t_0)$ be the corresponding change in the output x. Compute the first order Taylor approximation of x at the point t_0 to see that

$$x(t_0 + \delta t) \approx x(t_0) + x'(t_0)(\delta t) \tag{1}$$

from which you can then note that

$$x'(t_0) \approx \frac{\delta x}{\delta t} \tag{2}$$

(b) Define the *explicit Euler approximation sequence* $\{x_\tau\}_{\tau=0}^{T}$ so that $x(t_0) = x_0$ and at later times $x(t_0 + \tau \delta t) = x_\tau$. Show using the previous equations, the ODE itself, and the fact that $x_{\tau-1} + \delta x = x_\tau$ means can make a sequence

$$x_\tau = x_{\tau-1} + k x_{\tau-1} \delta t.$$

(c) Let $k = 1$, $\delta t = 0.01$, $t_0 = 0$, and let $x(0) = 1$. Plot the explicit Euler approximation sequence using the following URL `http://www.calcul.com/show/calculator/recursive`. Compare this graph to the solution $x(t) = e^t$.

(d) Let $k = -1$, $\delta t = 2$, $t_0 = 0$, and let $x(0) = 1$. Plot the approximation again. Is there something wrong? How does this compare to what the solution should be?