MATH 271, Homework 4, *Solutions*

Problem 1. Consider the following sequences,

$$a_n = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots,$$

and

$$b_n = 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \ldots, \frac{1}{n!}, \ldots.$$

(a) For what values of N do we need for $a_N < 0.01$ and $b_N < 0.01$? Note, these will be different values for N.

(b) Compute $\lim_{n\to\infty} a_n$.

(c) Compute $\lim_{n\to\infty} b_n$.

(d) Which sequence converges more quickly to its limit? (*Hint: consider a ratio of the terms of the sequences and take a limit. Part (a) should help you think about this.*)

Solution 1.

(a) Consider first the sequence given by a_n. Now, we want to find a value for N so that

$$a_N = \frac{1}{2^N} < 0.01.$$

We can play with this algebraically by

$$\begin{aligned} \frac{1}{2^N} &< 0.01 \\ 100 &< 2^N \\ \log_2(100) &< N \\ \approx 6.644 &< N. \end{aligned}$$

Since N is an integer, we round up to get $N = 7$.

For the sequence given by b_n we do the same and we want to find

$$\begin{aligned} b_N = \frac{1}{N!} &< 0.01 \\ 100 &< N! \end{aligned}$$

which has no nice inverse function to use (like $\log_2$ for 2^x), so we just need to try a few values. We have

$$1! = 1 \quad 2! = 2 \quad 3! = 6 \quad 4! = 24 \quad 5! = 120,$$

so $N = 5$ works.

(b) Since $a_n = f(n) = \frac{1}{2^n}$ we can consider

$$\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{2^x} = 0,$$

by our knowledge of limits from Calculus 1.

(c) We can do this a few ways. For one, we know that $n!$ grows unboundedly as n gets larger and larger so $\lim_{n\to\infty} n! = +\infty$ and so $\lim_{n\to\infty} \frac{1}{n!} = 0$. We could also prove this more rigorously by comparing b_n to a_n. Notice that for $K \geq 3$ we have $0 < b_K < a_K$ and since $a_n \to 0$, we have $b_n \to 0$. One could also use the ϵ definition for convergence.

(d) If we consider the limit of the ratio of the terms in the sequence as follows

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right|$$

then we can see which grows faster than the other. Let's investigate further

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{1}{2^n}}{\frac{1}{n!}} \right| \\ &= \lim_{n\to\infty} \left| \frac{n!}{2^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2} \right|. \end{aligned}$$

Here, notice that there are n terms in the numerator, and n in the denominator. However, most of the terms in the numerator are larger than 2 (and, for example, $4 = 2 \cdot 2$). So, the denominator is larger especially as n gets larger and we find

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = \infty$$

which means that $b_n \to 0$ faster than $a_n \to 0$.

Remark 1. I have shown some work here that looks complicated and we haven't totally covered in class. So, I don't expect your solutions to be as clean as mine! What I wanted you to do was just think about these ideas and try to gain intuition. Especially with part (d).

Problem 2. With the same a_n from 1, consider the series

$$A = \sum_{n=1}^{\infty} a_n.$$

(a) Write down the N^{th} partial sum A_N for this series.

(b) Does this sequence of partial sums converge? If so, to what?

(c) Note that this is an *geometric series* with $a = 1$ and $r = \frac{1}{2}$. However, we start from $n = 1$ instead of $n = 0$. Show the value that this series converges to using the formula for a geometric series.

Solution 2.

(a) The N^{th} partial sum is given by

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^N}.$$

(b) Let us take a look at the sequence of partial sums

$$A_1 = \frac{1}{2} \quad A_2 = \frac{3}{4} \quad A_2 = \frac{7}{8} \quad A_3 = \frac{15}{16},$$

which leads us to

$$A_N = \frac{2^N - 1}{2^N}.$$

Thus we can take

$$\lim_{N\to\infty} A_n = \lim_{N\to\infty} \frac{2^N - 1}{2^N} = 1.$$

So the series converges to 1.

(c) There is a formula for a geometric series

$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}.$$

If we take $a = 1$ and $r = \frac{1}{2}$ then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1 \cdot \frac{1}{2}}{1 - \frac{1}{2}} = \frac{1}{}.$$

So our result is identical.

Problem 3. With the same b_n from 1, consider the series

$$B = \sum_{n=0}^{\infty} b_n.$$

(a) Use the ratio test to show that this series converges.

(b) Approximate the value the series converges to by considering larger and larger partial sums.

(c) What number does this series converge to?

Solution 3.

(a) Consider the limit for the ratio test

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{1}{n+1} \right| \\
&= 0.
\end{aligned}$$

So by the ratio test, the series converges.

(b) Using a tool like WolframAlpha, we can compute approximations to the series using partial sums. For example, we can take

$$\begin{aligned}
\sum_{n=0}^{1} b_n &= 2 \\
\sum_{n=0}^{5} b_n &\approx 2.7166... \\
\sum_{n=0}^{50} b_n &\approx 2.718281828459
\end{aligned}$$

That is, I put

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sum[1/n!,{n,0,N}]
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into WolframAlpha (but of course replaced N with the chosen N values above).

(c) Again, one could use WolframAlpha to find what this series converges to and we get that

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718281828459.$$

Problem 4. Consider the *p-series*:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

(a) For $p = 1$, show that the ratio test is inconclusive.

(b) For $p = 2$, show that the ratio test is again inconclusive.

(c) Look up the sum of the series for $p = 1$ and $p = 2$. Notice how the ratio test is not perfect!

Solution 4.

(a) Let us take $p = 1$. Then we have

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

This is known as the *harmonic series*. Now, the limit for the ratio test is

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| &= \lim_{n\to\infty} \left| \frac{n}{n+1} \right| \\ &= 1. \end{aligned}$$

So the ratio test is inconclusive.

(b) Similarly, for $p = 2$, we have the limit for the ratio test

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| &= \lim_{n\to\infty} \left| \frac{n^2}{n^2+2n+1} \right| \\ &= \lim_{n\to\infty} \left| \frac{n^2}{n^2} \right| \\ &= 1. \end{aligned}$$

Note that here I used the fact that the leading power term dominates in the limit. Thus, we again have the ratio test is inconclusive.

(c) We have that the p-series for $p = 1$ diverges and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$