MATH 271, Exam 3

Oral Examination Problems

Due one hour before your exam time slot.

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page. You can use online tools such as Desmos and Wolfram Alpha to check your work, but you will need to explain how you arrived at your answers. You can work with other students and this is, in fact, encouraged! However, I will not be giving out direct help for these problems but can answer questions about previous problems and notes, for example. Ambiguous or illegible answers will not be counted as correct. Scan your solutions and submit them as a pdf on Canvas under Oral Exam 3.

Note, there are four total problems and a bonus.

Problem 1. Consider the following vectors in $\mathbb{R}^2$:

$$\vec{\boldsymbol{u}} = \hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}} \quad \vec{\boldsymbol{v}} = -2\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} \quad \vec{\boldsymbol{w}} = -\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}.$$

(a) Draw all vectors $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{w}}$ in the plane. Draw $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$ in the plane as well.
(b) Are any of these vectors orthogonal? Explain.
(c) Compute the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$. Draw this parallelogram in the plane.
(d) Explain why $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ form a basis for $\mathbb{R}^2$.
(e) Given the vector $\vec{\boldsymbol{y}} = 13\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}$, write $\vec{\boldsymbol{y}}$ as a linear combination of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

Problem 2. Consider a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{aligned} T(\hat{\boldsymbol{e}}_1) &= \hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3 \\ T(\hat{\boldsymbol{e}}_2) &= \hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 \\ T(\hat{\boldsymbol{e}}_3) &= 2\hat{\boldsymbol{e}}_1 + 2\hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3 \end{aligned}$$

(a) What is the kernel of T?
(b) What is the image of T?
(c) Write down a matrix representation $[T]$ for the transformation T.
(d) Compute the characteristic polynomial for T, $p(\lambda)$. Then show that $p([T]) = [0]$ is the zero matrix. *This just means to plug in the matrix $[T]$ for the variable λ and $[0]$ is the all zeros matrix.*

Problem 3. Take the matrix

$$[A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Show that $\hat{\boldsymbol{e}}_1$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

(b) Compute $\det([A])$ and $\operatorname{tr}([A])$ and using these quantities plus your knowledge from (a), show that the other two eigenvalues are $\lambda_2 = 1$ and $\lambda_3 = -1$. (DO NOT USE THE CHARACTERISTIC POLYNOMIAL!)

Problem 4. Consider the matrix

$$[A] = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors for this matrix.

(b) Construct the matrix $[P]$ such that

$$[\Lambda] = [P]^{-1}[A][P]$$

from the eigenvectors you found.

(c) Find $[P]^{-1}$ and compute

$$[\Lambda] = [P]^{-1}[A][P].$$

Is this $[\Lambda]$ diagonal?

Problem 5 (BONUS)**.** Let V be a vector space and let $\langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle$ be an inner product. We say that an operator $T \colon V \to V$ is self-adjoint if $\langle T\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle = \langle \vec{\boldsymbol{u}}, T\vec{\boldsymbol{v}} \rangle$. Let $T \colon \mathbb{C}^n \to \mathbb{C}^n$ be self adjoint and take the inner product on $\mathbb{C}^n$ to be given by

$$\langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle = \sum_{j=1}^{n} u_j v_j^*$$

We want to prove the two theorems in the text.

(a) Show that all eigenvalues of A are real.

(b) Show that eigenvectors corresponding to different eigenvalues are orthogonal with the hermitian inner product.