MATH 271, Exam 2

Oral Examination Problems

Due one hour before your exam time slot.

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page. You can use online tools such as Desmos and Wolfram Alpha to check your work, but you will need to explain how you arrived at your answers. You can work with other students and this is, in fact, encouraged! However, I will not be giving out direct help for these problems but can answer questions about previous problems and notes, for example. Ambiguous or illegible answers will not be counted as correct. Scan your solutions and submit them as a pdf on Canvas under Oral Exam 2.

Note, there are three total problems.

Problem 1. Sequences show up in the realm of approximation all the time. One of the first sequences one may encounter comes from *Newton's method* for calculating roots to a function (see this week's discussion for more). The idea is that we are handed a function $f(x)$ and pick an initial guess x_0 as a candidate for a root to $f(x)$. Then, we can compute a tangent line approximation to $f(x)$ at the point x_0. That is,

$$f(x) \approx f'(x_0)(x - x_0) + f(x_0).$$

From there, we can see where this tangent line intersects the $y = 0$ axis which gives us a new x-value we call x_1. If we solve for x_1, we find that we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

From here, we can generate a recursive sequence given by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

that *can possibly* give a better and better approximation to a root of $f(x)$, meaning that x_n will tend to be an better approximation to a root of $f(x)$ than x_{n-1}. This method is in fact exactly how your TI-XX calculator finds zeros to a function (notice that you give bounds and an initial guess when you do this)!

Feel free to use a calculator for this problem.

(a) Consider the function $f(x) = x^2 - 1$ with the starting point of $x_0 = 2$, draw a picture that illustrates the first two iterations of Newton's method. (*Hint: This requires drawing a tangent line to $f(x)$ at two different points and telling me what the intersections of these lines with the x-axis represent.*)

(b) Write the first four terms in the sequence for Newton's method. That is, find $x_0, \ldots, x_3$ and write

$$\{x_n\}_{n=0}^{\infty} = x_0,\ x_1,\ x_2,\ x_3, \ldots.$$

(c) This sequence approaches $+\sqrt{1} = 1$ since this is the nearest root to $f(x)$ compared to our starting value. What is the first value of N where the error $|x_N - 1| < 10^{-6}$?

(d) If we instead have $g(x) = x^3 - 2x + 2$ with an initial guess of $x_0 = 0$, write the first four terms of the sequence $x_0, \ldots, x_3$ for Newton's method. Does this sequence seem to approach the real root $x \approx -1.7693$?

Problem 2. Bessel functions can arise as the vibrational states of disk shaped objects (e.g., ripples that form in drum heads and cymbals when struck), or as electromagnetic waves in a cylindrical waveguide, or in diffraction from DNA. The *Bessel equation of order zero* is given by

$$x^2 f'' + x f' + x^2 f = 0,$$

where f is a function of x.

These parts will walk you through finding a solution to the equation. I give you the answer in part (a) (which you will check is a solution), but you can (and should) use this to help guide you through the remaining parts. *The trick for this problem is to be careful and to write out the first few terms of series when needed. Wolfram Alpha can help you write partial sums.*

(a) Show that the power series

$$f(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n},$$

is a general solution to the Bessel equation of order zero.

(b) Take a power series ansatz for $f(x)$. Then, using that ansatz in the ODE, show that you arrive at the condition

$$a_1 x + \sum_{n=2}^{\infty} \left[n^2 a_n + a_{n-2} \right] x^n = 0.$$

(c) Argue now that all the odd terms of the series for your solution must be odd. Do not use the given answer in (a) as your reasoning!

(d) Determine a general form for a_{2n} and show that you get the same solution as in part (a).

Problem 3. Consider the two example systems from quantum mechanics. First, for a particle in a box of length 1 we have the equation

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi,$$

with boundary conditions $\Psi(0) = 0$ and $\Psi(1) = 0$.

Second, the Quantum Harmonic Oscillator (QHO)

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right)\Psi = E\Psi$$

(a) Write down the states for both systems. What are their similarities and differences?
(b) Write down the energy eigenvalues for both systems. What are their similarities and differences?
(c) Plot the first three states of the QHO along with the potential for the system.
(d) Explain why you can observe a particle outside of the "classically allowed region". *Hint: you can use any state and compute an integral to determine a probability of a particle being in a given region.*