MATH 271, Homework 8, *Solutions*

Problem 1. Let a mass m_1 weighing $1kg.$ be placed at $\vec{\boldsymbol{r}}_1 = 2\hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$ and a mass m_2 of $2kg.$ be placed at $\vec{\boldsymbol{r}}_2 = 4\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}$. Where must a mass m_3 of $3kg.$ be placed so that the center of mass is at the origin $\vec{\boldsymbol{0}}$?

Solution 1. One can compute the center of mass $\vec{\boldsymbol{R}}_{CM}$ by

$$\vec{\boldsymbol{R}}_{CM} = \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + m_3\vec{\boldsymbol{r}}_3}{m_1 + m_2 + m_3}.$$

Here, we know everything but $\vec{\boldsymbol{r}}_3$. Since we want the center of mass at the origin $\vec{\boldsymbol{0}}$, then

$$\begin{aligned}\vec{\boldsymbol{0}} &= \frac{1}{m_1 + m_2 + m_3}\left(m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + m_3\vec{\boldsymbol{r}}_3\right)\\ \begin{pmatrix}0\\0\\0\end{pmatrix} &= \frac{1}{6}\left(\begin{pmatrix}2\\-3\\-1\end{pmatrix} + 2\begin{pmatrix}0\\4\\-2\end{pmatrix} + 3\begin{pmatrix}x\\y\\z\end{pmatrix}\right).\end{aligned}$$

What we have above is three equations and three unknowns. That is, one equation for the $\hat{\boldsymbol{x}}$-component, one for the $\hat{\boldsymbol{y}}$-component, and one for the $\hat{\boldsymbol{z}}$-component. We have

$$\begin{aligned}0 &= \frac{1}{6}(2 + 2\cdot 0 + 3x)\\ 0 &= \frac{1}{6}(-3 + 2\cdot 4 + 3y)\\ 0 &= \frac{1}{6}(-1 + 2\cdot(-2) + 3z).\end{aligned}$$

Taking the first, we find

$$\begin{aligned}0 &= \frac{1}{3} + \frac{1}{2}x\\ -\frac{1}{3} &= \frac{1}{2}x\\ \implies \quad x &= -\frac{2}{3}.\end{aligned}$$

Next,

$$\begin{aligned}0 &= -\frac{1}{2} + \frac{4}{3} + \frac{1}{2}y\\ -\frac{5}{6} &= \frac{1}{2}y\\ \implies \quad y &= -\frac{5}{3}.\end{aligned}$$

Lastly, we have

$$\begin{aligned} 0 &= -\frac{1}{6} - \frac{2}{3} + \frac{1}{2}z \\ \frac{5}{6} &= \frac{1}{2}z \\ \implies \quad z &= \frac{5}{3}. \end{aligned}$$

Thus we have that $\vec{\boldsymbol{r}}_3 = -\frac{2}{3}\hat{\boldsymbol{x}} - \frac{5}{3}\hat{\boldsymbol{y}} + \frac{5}{3}\hat{\boldsymbol{z}}$.

Problem 2. Which of the following are linear transformations? For those that are not, which properties of *linearity* (the properties (i) and (ii) in our notes) fail? Show your work.

(a) $T_a \colon \mathbb{R} \to \mathbb{R}$ given by $T_a(x) = \frac{1}{x}$.

(b) $T_b \colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T_b \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(c) $T_c \colon \mathbb{R} \to \mathbb{R}^3$ given by

$$T_c(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}.$$

(d) $T_d \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T_d \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix}.$$

Solution 2.

(a) This transformation fails both properties. For (i), take

$$T_a(x+y) = \frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y} = T_a(x) + T_a(y).$$

For (ii), take

$$T_a(\alpha x) = \frac{1}{\alpha x} \neq \alpha \frac{1}{x} = \alpha T_a(x).$$

(b) This is a linear transformation. To see (i) holds, take

$$\begin{aligned} T_b(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) &= T_b \left(\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \right) \\ &= T_b \begin{pmatrix} u_x + v_x \\ u_y + v_y \\ u_z + v_z \end{pmatrix} \\ &= \begin{pmatrix} u_x + v_x \\ u_y + v_y \end{pmatrix} \\ &= \begin{pmatrix} u_x \\ u_y \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ &= T_b(\vec{\boldsymbol{u}}) + T_b(\vec{\boldsymbol{v}}). \end{aligned}$$

And for (ii), we take

$$\begin{aligned} T_b(\alpha\vec{\boldsymbol{v}}) &= T_b\left(\alpha\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}\right) \\ &= T_b\begin{pmatrix} \alpha v_x \\ \alpha v_y \\ \alpha v_z \end{pmatrix} \\ &= \begin{pmatrix} \alpha v_x \\ \alpha v_y \end{pmatrix} \\ &= \alpha\begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ &= \alpha T_b(\vec{\boldsymbol{v}}). \end{aligned}$$

(c) This is not a linear transformation as both properties fail. Indeed, for (i) we take

$$\begin{aligned} T_c(\vec{\boldsymbol{u}}+\vec{\boldsymbol{v}}) &= T_c\left(\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}\right) \\ &= T_c\begin{pmatrix} u_x+v_x \\ u_y+v_y \\ u_z+v_z \end{pmatrix} \\ &= \begin{pmatrix} u_x+v_x \\ (u_y+v_y)^2 \\ (u_z+v_z)^3 \end{pmatrix}, \end{aligned}$$

whereas

$$\begin{aligned} T_c(\vec{\boldsymbol{u}})+T_c(\vec{\boldsymbol{v}}) &= T_c\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + T_c\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \\ &= \begin{pmatrix} u_x \\ u_y^2 \\ u_z^3 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y^2 \\ v_y^3 \end{pmatrix} \\ &= \begin{pmatrix} u_x+v_x \\ u_y^2+v_y^2 \\ u_z^3+v_z^3 \end{pmatrix}. \end{aligned}$$

Note that $u_y^2+v_y^2 \neq (u_y+v_y)^2$ and $u_z^3+v_z^3 \neq (u_z+v_z)^3$.

To see that (ii) does not hold, take

$$\begin{aligned}
T_c(\alpha\vec{\boldsymbol{v}}) &= T_c\begin{pmatrix}\alpha v_x\\ \alpha v_y\\ \alpha v_z\end{pmatrix}\\
&= \begin{pmatrix}\alpha v_x\\ \alpha^2 v_y^2\\ \alpha^3 v_z^3\end{pmatrix},
\end{aligned}$$

whereas

$$\alpha T_c(\vec{\boldsymbol{v}}) = \begin{pmatrix}\alpha v_x\\ \alpha v_y^2\\ \alpha v_z^3\end{pmatrix}.$$

These are clearly not equal for every scalar α.

(d) This function is linear. For (i), we have

$$\begin{aligned}
T_d(\vec{\boldsymbol{u}}+\vec{\boldsymbol{v}}) &= T_d\begin{pmatrix}u_x+v_x\\ u_y+v_y\end{pmatrix}\\
&= \begin{pmatrix}(u_x+v_x)+(u_y+v_y)\\ (u_x+v_x)+(u_y+v_y)\\ (u_x+v_x)+(u_y+v_y)\end{pmatrix}\\
&= \begin{pmatrix}u_x+u_y\\ u_x+u_y\\ u_x+u_y\end{pmatrix} + \begin{pmatrix}v_x+v_y\\ v_x+v_y\\ v_x+v_y\end{pmatrix}\\
&= T(\vec{\boldsymbol{u}}) + T(\vec{\boldsymbol{v}}).
\end{aligned}$$

And for (ii) we have

$$\begin{aligned}
T_d(\alpha\vec{\boldsymbol{v}}) &= T_d\begin{pmatrix}\alpha v_x\\ \alpha v_y\end{pmatrix}\\
&= \begin{pmatrix}\alpha v_x+\alpha v_y\\ \alpha v_x+\alpha v_y\\ \alpha v_x+\alpha v_y\end{pmatrix}\\
&= \alpha T_d(\vec{\boldsymbol{v}}).
\end{aligned}$$

Problem 3. Consider the linear transformation $J\colon R^2 \to \mathbb{R}^2$ defined by

$$J(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}} \qquad \text{and} \qquad J(\hat{\boldsymbol{y}}) = -\hat{\boldsymbol{x}}.$$

This linear transformation is fundamental in understanding how we can reconstruct complex numbers using matrices.

(a) Show that $J^2 = J \circ J = -1$.

(b) Determine a matrix representation for J and denote it by $[J]$.

(c) Recall that we can represent a complex number as $z = x + iy$ and that we can represent z as a vector in $\mathbb{R}^2$ as $\vec{\boldsymbol{\zeta}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$. Show that $J\vec{\boldsymbol{\zeta}}$ corresponds to iz.

(d) We can completely reconstruct a representation of $\mathbb{C}$ by using a matrix representation. In particular, we can take

$$[z] = x[I] + y[J].$$

Show that we recover the complex addition and multiplication using this representation.

(e) We can represent a unit complex number as $z = e^{i\theta}$. Show that the representation described before leads to

$$[z] = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Solution 3. (a) Let $\vec{\boldsymbol{v}} = v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}}$ be some arbitrary vector in $\mathbb{R}^2$. Then,

$$\begin{aligned} J^2(\vec{\boldsymbol{v}}) = J(J(\vec{\boldsymbol{v}})) &= J(J(v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}})) \\ &= J(v_1J(\hat{\boldsymbol{x}}) + v_2J(\hat{\boldsymbol{y}})) \\ &= J(v_1\hat{\boldsymbol{y}} - v_2\hat{\boldsymbol{x}}) \\ &= v_1J(\hat{\boldsymbol{y}}) - v_2J(\hat{\boldsymbol{x}}) \\ &= -v_1\hat{\boldsymbol{x}} - v_2\hat{\boldsymbol{y}} \\ &= -\vec{\boldsymbol{v}}. \end{aligned}$$

So, yes, J^2 acts like scaling by -1.

(b) We determine a matrix for J by using the definition of J on $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. In particular,

$$[J] = \begin{pmatrix} | & | \\ J(\hat{\boldsymbol{x}}) & J(\hat{\boldsymbol{y}}) \\ | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

One can check here that $[J]^2 = -[I]$, where $[I]$ is the identity matrix. This confirms that $[J]$ satisfies the relationship we saw in (a).

(c) In the complex plane, we let $z = x + iy$ and we can note that

$$iz = -y + ix.$$

Now, we can think of z as a vector in $\mathbb{R}^2$ by noticing that the vector $\vec{\boldsymbol{\zeta}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ corresponds to the same exact point geometrically. Then, if we apply J we have

$$J\vec{\boldsymbol{\zeta}} = -y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}},$$

which is exactly how z was transformed when we multiplied by i. Keep in mind that i rotates a complex number z by $\pi/2$ in the counterclockwise direction and J does the same to vectors $\vec{\boldsymbol{\zeta}}$. To see this most fully, consider drawing a picture of both transformations.

(d) In the complex plane, we can take two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we have

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \qquad \text{and} \qquad z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Notice that addition is componentwise and keep track of this result from the multiplication.

Now, we can consider two matrices $[z_1] = x_1[I] + y_1[J]$ and $[z_2] = x_2[I] + y_2[J]$ and see what we get through addition and multiplication. We have

$$[z_1] + [z_2] = (x_1 + x_2)[I] + (y_1 + y_2)[J].$$

This is due to how matrices add componentwise and we can see that this corresponds to the addition in $\mathbb{C}$. Next, we have

$$\begin{aligned}
[z_1][z_2] &= (x_1[I] + y_1[J])(x_2[I] + y_2[J]) \\
&= x_1x_2[I]^2 + y_1x_2[J][I] + x_1y_2[I][J] + y_1y_2[J]^2 \\
&= x_1x_2[I] + y_1x_2[J] + x_1y_2[J] - y_1y_2[I] \\
&= (x_1x_2 - y_1y_2)[I] + (x_1y_2 + x_2y_1)[J].
\end{aligned}$$

Note that I use the facts $[J][I] = [I][J] = [J]$, $[I]^2 = [I]$, and from (a) we know $[J]^2 = -[I]$. If we take a look at the end result, we can see that this is the same multiplication result as z_1z_2 in $\mathbb{C}$.

(e) Using our knowledge from the previous problem, and Euler's formula, we know that we can take

$$[e^{i\theta}] = \cos(\theta)[I] + \sin(\theta)[J].$$

Writing out the matrices explicitly yields

$$[e^{i\theta}] = \begin{pmatrix} \cos(\theta) & 0 \\ 0 & \cos(\theta) \end{pmatrix} + \begin{pmatrix} 0 & -\sin(\theta) \\ \sin(\theta) & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

as intended.

Remark 1. If one goes to look up a rotation matrix for $\mathbb{R}^2$, you will find the matrix you found in (e). So, this goes to show that complex arithmetic captures rotations nicely through Euler's formula. Moreover, the matrix representation for a complex number is faithful in describing all that we need.

Problem 4. Write down the matrix for the following linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}.$$

Solution 4. We need that

$$[T]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}$$

via matrix multiplication. Since the input vector is a 3-dimensional vector, and the output vector is 3-dimensional, we must have that $[T]$ is a 3×3-matrix. Hence,

$$[T] = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t_{11}x + t_{12}y + t_{13}z \\ t_{21}x + t_{22}y + t_{23}z \\ t_{31}x + t_{32}y + t_{33}z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}.$$

If we match the coefficients on the x, y, and z, we find that

$$[T] = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$

Problem 5. Take the following matrices:

$$[A] = \begin{pmatrix} 4 & 3 & 10 & 2 \\ 1 & 1 & 0 & 9 \end{pmatrix}, \quad [B] = \begin{pmatrix} 8 & 5 & 8 \\ 10 & 9 & 2 \\ 4 & 6 & 3 \end{pmatrix}, \quad [C] = \begin{pmatrix} 0 & 0 & 9 \\ 7 & 9 & 9 \\ 1 & 9 & 9 \\ 3 & 3 & 1 \end{pmatrix}$$

(a) Compute either $[A][C]$ or $[C][A]$ and state which multiplication is not possible.

(b) Compute either $[B][C]$ or $[C][B]$ and state which multiplication is not possible.

(c) Can you add any of these matrices?

(d) Describe each matrix as linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$. What is m and n for each? How does this relate to the number of rows and columns?

Solution 5.

(a) The matrix $[A]$ is a 2×4 matrix and matrix $[C]$ is a 4×3 matrix. So we can compute $[A][C]$ but not $[C][A]$. Given that, we also expect the output to be a 2×3 matrix. So, we have

$$[A][C] = \begin{pmatrix} 4 & 3 & 10 & 2 \\ 1 & 1 & 0 & 9 \end{pmatrix} \begin{pmatrix} 0 & 0 & 9 \\ 7 & 9 & 9 \\ 1 & 9 & 9 \\ 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 37 & 123 & 155 \\ 34 & 36 & 27 \end{pmatrix}.$$

(b) $[B]$ is a 3×3 matrix so we can take $[C][B]$ but not $[B][C]$. We get

$$[C][B] = \begin{pmatrix} 0 & 0 & 9 \\ 7 & 9 & 9 \\ 1 & 9 & 9 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 8 & 5 & 8 \\ 10 & 9 & 2 \\ 4 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 36 & 54 & 27 \\ 182 & 170 & 101 \\ 134 & 140 & 53 \\ 58 & 48 & 33 \end{pmatrix}.$$

(c) We can always add a matrix to itself, so, for example $[A]+[A]$, $[B]+[B]$, and $[C]+[C]$ make sense. However, since the dimensions of $[A]$, $[B]$, and $[C]$ all differ, we cannot add in any other way.

(d) The number of columns of a matrix denotes the input dimension m, and the number of rows denotes the output dimension n. So

$$A\colon \mathbb{R}^4 \to \mathbb{R}^2, \quad B\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad C\colon \mathbb{R}^3 \to \mathbb{R}^4.$$

Problem 6. Solve the following equation.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 11 \end{pmatrix}.$$

Solution 6. First, we create the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 1 & 8 \\ 1 & 2 & 2 & 11 \end{array}\right).$$

We can subtract R1 from both R2 and R3 to get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \end{array}\right).$$

Then subtract R3 from R1 to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 5 \end{array}\right).$$

Finally, subtract R2 from R3 to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

This yields our result in the right most column in that $x = 1$, $y = 2$, and $z = 3$.