MATH 271, Homework 7, *Solutions*

Due November 1st

Problem 1. Let S be the set of general solutions $x(t)$ to the following homogeneous linear differential equation

$$x'' + f(t)x' + g(t)x = 0.$$

Show that this set S is a vector space over the complex numbers by doing the following. Let $x(t), y(t), z(t) \in S$ be solutions to the above equation and let $\alpha, \beta \in \mathbb{C}$ be complex scalars.

(a) Write down the eight requirements for S to be a vector space.

(b) Identify the $\vec{\mathbf{0}} \in S$ and $1 \in \mathbb{C}$.

(c) Show that $\alpha x(t) + \beta y(t) \in S$. That is, show that a superposition of solutions is also a solution. *Hint: We have shown this before.*

Solution 1. (a) We can remember these requirements via the acronym CANI ADDU. So we have for the vector addition properties

- Commutivity: If we have two solutions $x(t)$ and $y(t)$ in the set S, then we know

$$x(t) + y(t) = y(t) + x(t)$$

is satisfied.

- Associativity: If we have three solutions $x(t), y(t), z(t) \in S$, then we know

$$(x(t) + y(t)) + z(t) = x(t) + (y(t) + z(t))$$

is satisfied.

- Neutral Element: We have that there exists the zero function $0 \in S$ such that

$$0 + x(t) = x(t).$$

- Inverses: Given an $x(t) \in S$, we have the function $-x(t) \in S$ such that

$$x(t) + (-x(t)) = 0.$$

Then we have the scalar multiplication properties

- Associativity: If we have $\alpha, \beta \in \mathbb{C}$ and $x(t) \in S$ then we have

$$\alpha(\beta x(t)) = (\alpha\beta)x(t)$$

holds.

- Distribution: Given $\alpha, \beta \in \mathbb{C}$ and $x(t) \in S$ we have

$$(\alpha + \beta)x(t) = \alpha x(t) + \beta x(t)$$

holds.

- Distribution: Given $\alpha \in \mathbb{C}$ and $x(t), y(t) \in S$, we have

$$\alpha(x(t) + y(t)) = \alpha x(t) + \alpha y(t)$$

holds.

- Unit element: We have $1 \in \mathbb{C}$ satisfies that for any $x(t) \in S$ that

$$1x(t) = x(t).$$

(b) Now, note that above we defined $\vec{\mathbf{0}} \in S$ to be the zero function 0. That is, the function that is 0 for every value of t. Note that 0 is a solution to the equation since

$$0'' + f(t) \cdot 0' + g(t)0 = 0.$$

Then, we have $1 \in \mathbb{C}$ that satisfies the necessary property too. In this case, 1 is literally the unit element we care about.

Remark 1. Not all vector spaces will have such obvious neutral elements, $\vec{\mathbf{0}}$. Likewise, not all fields will have an obvious unit element 1. This is why we must be a bit careful at times.

(c) Now, the biggest requirement for a vector space is that linear combinations of vectors actually produce another vector. This is quite obvious in the plane $\mathbb{R}^2$ for example, but here, it is not necessarily obvious.

Now, we take $\alpha, \beta \in \mathbb{C}$ and $x(t), y(t) \in S$ and we consider the linear combination

$$z(t) = \alpha x(t) + \beta y(t).$$

We then wish to show that this linear combination (or superposition) is a solution as well. So we plug in $z(t)$ into our equation as follows

$$\begin{aligned} z''(t) + f(t)z'(t) + g(t)z(t) &= (\alpha x''(t) + \beta y''(t)) + f(t)(\alpha x'(t) + \beta y'(t)) + g(t)(\alpha x(t) + \beta y(t)) \\ &= \alpha \left[x''(t) + f(t)x'(t) + g(t)x(t)\right] + \beta \left[y''(t) + f(t)y'(t) + g(t)y(t)\right] \\ &= 0, \end{aligned}$$

since we knew that $x(t)$ and $y(t)$ themselves are solutions. Thus, $z(t)$ is as well and now we have that S is a vector space.

Problem 2. Consider the following vectors in the real plane $\mathbb{R}^2$. We let

$$\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -3\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}}.$$

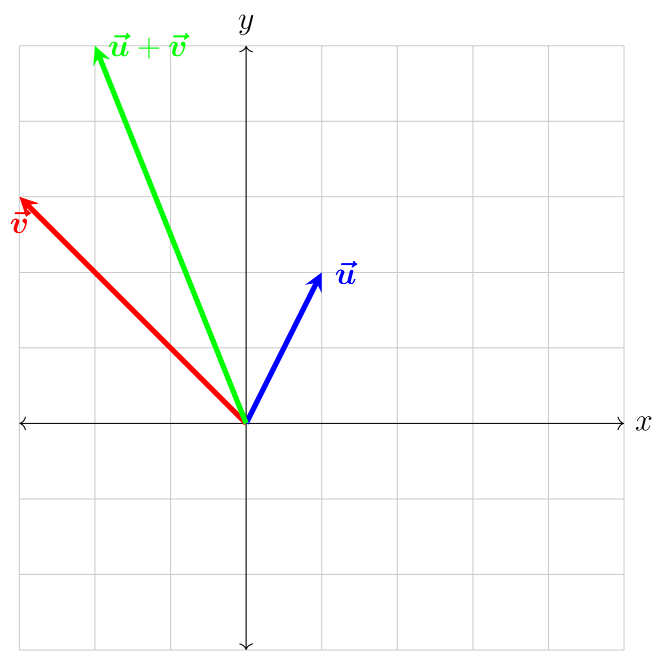
(a) Draw both $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ in the plane and label the origin.

(b) Draw the vector $\vec{\boldsymbol{w}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$ in the plane.

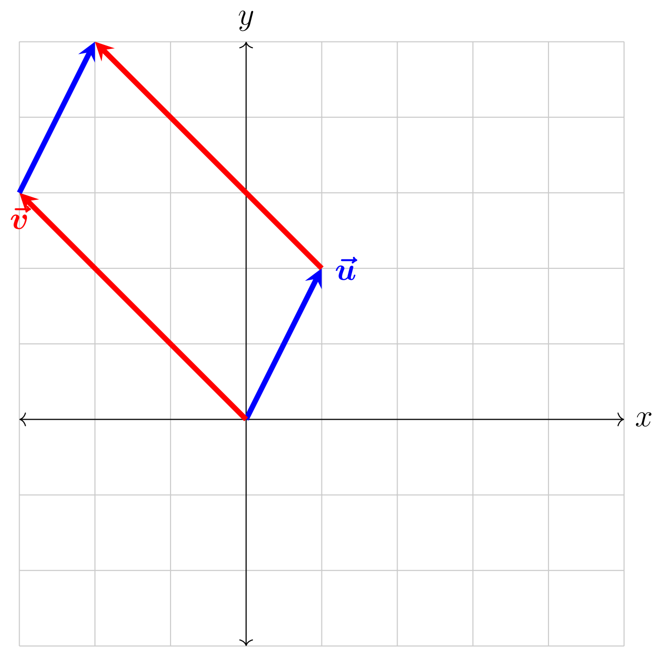
(c) Find the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

Solution 2. (a) See the plane below.

(b) Both (a) and (b) are in the plane here:



(c) One could compute the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ in many ways. First, let us see what this looks like:



In order to compute this area, we can use the cross product by thinking of these vectors as being in 3-dimensional space by

$$\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 0\hat{\boldsymbol{z}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -3\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} + 0\hat{\boldsymbol{z}}.$$

Then the cross product of these two vectors must only have a z-component since these two vectors lie in the $xy-$plane. Thus, we can compute

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = (1 \cdot 3 - (-3) \cdot 2)\hat{\boldsymbol{z}} = 9\hat{\boldsymbol{z}}.$$

Hence, the area is $\|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}\| = \|9\hat{\boldsymbol{z}}\| = 9$.

Problem 3.

(a) We can reflect a vector in the plane by first reflecting basis vectors. Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function be defined by

$$R(\hat{\boldsymbol{x}}) = -\hat{\boldsymbol{x}} \qquad \text{and} \qquad R(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{y}}.$$

Let $\vec{\boldsymbol{v}} = \alpha_1\hat{\boldsymbol{x}} + \alpha_2\hat{\boldsymbol{y}}$ and define

$$R(\vec{\boldsymbol{v}}) = \alpha_1 R(\hat{\boldsymbol{x}}) + \alpha_2 R(\hat{\boldsymbol{y}}).$$

When this is the case, we call the function T <u>linear</u>.
Show that R reflects the vector $\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}}$ about the y-axis and draw a picture.

(b) We can rotate a vector in the plane by first rotating the basis vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Define a <u>linear</u> function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}} \qquad \text{and} \qquad T(\hat{\boldsymbol{y}}) = -\hat{\boldsymbol{x}}.$$

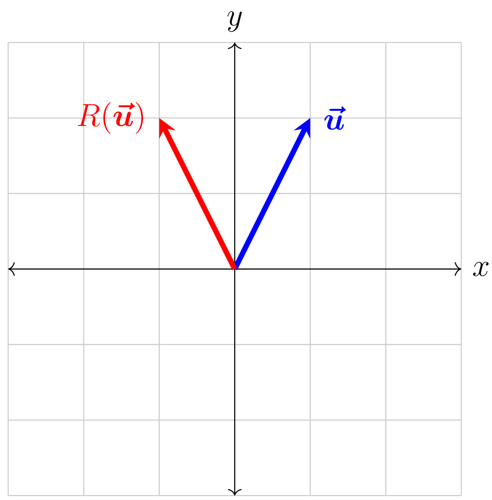
Show that T rotates $\vec{\boldsymbol{u}}$ by $\pi/2$ in the counterclockwise direction and draw a picture.

Solution 3.

(a) So, we can take the vector $\vec{\boldsymbol{u}}$ and then we have

$$R(\vec{\boldsymbol{u}}) = 1R(\hat{\boldsymbol{x}}) + 2R(\hat{\boldsymbol{y}}) = -1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}}.$$

So we can plot both $\vec{\boldsymbol{u}}$ and $R(\vec{\boldsymbol{u}})$ in the plane:

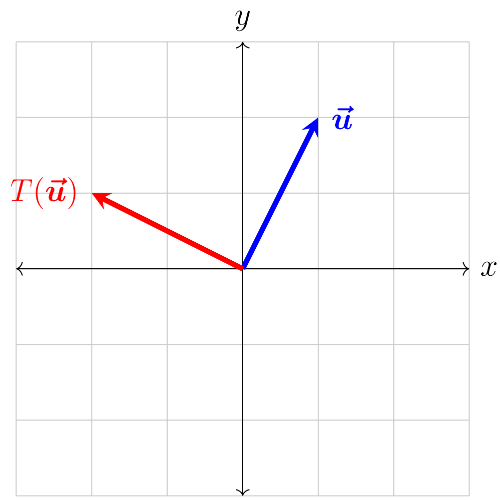


We can see that this is definitely the reflection of the vector $\vec{\boldsymbol{u}}$ across the y-axis.

(b) We can now do this for the function T to get

$$T(\vec{\boldsymbol{u}}) = 1T(\hat{\boldsymbol{x}}) + 2T(\hat{\boldsymbol{y}}) = 1\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{x}}.$$

Then we can plot both $\vec{\boldsymbol{u}}$ and $T(\vec{\boldsymbol{u}})$ in the plane:



We can see that this is definitely the rotation of the vector $\vec{u}$ an angle of $\pi/2$ in the counter-clockwise direction.

Problem 4. Consider the following vectors in space $\mathbb{R}^3$

$$\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -2\hat{\boldsymbol{x}} + 1\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}.$$

(a) Compute the dot product $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$.

(b) Compute the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$.

(c) Compute the lengths $\|\vec{\boldsymbol{u}}\|$ and $\|\vec{\boldsymbol{v}}\|$ using the dot product.

(d) Compute the angle between vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

(e) Compute the projection of $\vec{\boldsymbol{u}}$ in the direction of $\vec{\boldsymbol{v}}$.

Solution 4.

(a) We have that

$$\begin{aligned}\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} &= 1 \cdot (-2) + 2 \cdot 1 + 3 \cdot (-3)\\ &= -6.\end{aligned}$$

(b) Here, feel free to use a formula for a cross product instead of writing it all out. We will find that

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = -7\hat{\boldsymbol{x}} - 4\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}.$$

(c) We compute the lengths using the dot product by

$$\|\vec{\boldsymbol{u}}\| = \sqrt{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

Likewise

$$\|\vec{\boldsymbol{v}}\| = \sqrt{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}} = \sqrt{(-2)^2 + 1^2(-2)^2} = \sqrt{9} = 3.$$

(d) We can find the angle θ between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ with the information we already have. Indeed, recall the formula

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \|\vec{\boldsymbol{u}}\|\|\vec{\boldsymbol{v}}\| \cos\theta.$$

In the previous parts, we have computed $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = -6$, $\|\vec{\boldsymbol{u}}\| = \sqrt{14}$, and $\|\vec{\boldsymbol{v}}\| = 3$ so we now just have the following

$$-6 = 3\sqrt{14}\cos\theta.$$

Thus,

$$\theta = \arccos\left(\frac{-6}{3\sqrt{14}}\right) \approx 2.1347 \text{ radians}.$$

One could also use the fact that $\|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}\| = \|\vec{\boldsymbol{u}}\|\|\vec{\boldsymbol{v}}\| \sin\theta$ since we had already compute the cross product earlier.

(e) The projection of $\vec{\boldsymbol{u}}$ in the direction of $\vec{\boldsymbol{v}}$ is simply asking for how much of the vector $\vec{\boldsymbol{u}}$ is in the direction of $\vec{\boldsymbol{v}}$. One can arrive at this purely through trigonometry, but we have the dot product at our disposal. The normalized vector $\hat{\boldsymbol{v}}$ points in the direction of $\vec{\boldsymbol{v}}$ with length 1 and

$$\hat{\boldsymbol{v}} = \frac{1}{\|\vec{\boldsymbol{v}}\|}\vec{\boldsymbol{v}} = \frac{1}{3}\vec{\boldsymbol{v}}.$$

Then, the projection can be computed by

$$\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} = \frac{1}{3}\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = -2.$$

One should attempt to recover this notion by doing some trigonometry.