MATH 271, Homework 5

Due October 9^{th}

Problem 1. p-series are actually related to a very important function called the *Riemann zeta function*. This function is involved in a million dollar math problem! If you're interested in other million dollar problems, look up the Clay Institute Millennium Problems. The Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(a) Use the integral test to show that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. Look up what this series converges to and write it down. This is $\zeta(2)$.

(b) Use the comparison test to show that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges. This converges as well to $\zeta(3)$. Look up what this approximate value is.

Problem 2. Find the radius of convergence for the following power series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$;

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

Problem 3. Consider the two series

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{and} \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

(a) Show that $\cos(-x) = \cos(x)$.

(b) Show that $\sin(-x) = -\sin(x)$.

(c) To take a derivative of a power series $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ we can do the following:

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \frac{d}{dx} x^n.$$

Compute $\frac{d}{dx}\sin(x)$ and $\frac{d}{dx}\cos(x)$ and show that they are equal to what you already know. *Warning: be careful with the powers of x in the case with* $\sin$ *and* $\cos$*!*

Problem 4. Consider the function

$$f(x) = \frac{1}{1-x}.$$

(a) Compute the Taylor series centered at $a = 0$ for the function.

(b) Find the antiderivative $\int \frac{dx}{1-x}$ using the Taylor series for $f(x)$ found in (a).

(c) Write down the Taylor series for $\ln(1-x)$ centered at $a = 0$ and compare to your answer in (b).

Problem 5. (a) Compute the Taylor series centered at $a = 0$ for $f(x) = e^{-\frac{x^2}{2}}$.

(b) Use the Taylor series for e^x and modify it to find a power series for $f(x)$. Is this the same as the series in (a)?

(c) Plot the original function $f(x)$ compared to the first, second, third, and fourth term approximation for the series on the same graph.

Problem 6. How can we approximate a (possibly complicated) function by using a power series? Why is this useful (specifically for computation on a computer)?