MATH 271, Exam 3

Take Home Portion

Due December 4$^{\text{th}}$ at the start of class

Name ____

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page, but no other online resources (including calculators or WolframAlpha). for this portion of the exam. **Do not discuss any problem any other person.** All of your solutions should be easily identifiable and supporting work must be shown. Ambiguous or illegible answers will not be counted as correct. **Print out this sheet and staple your solutions to it. Use a new page for each problem.**

Problem 1 _____/15 **Problem 2** _____/10

Note, these problems span two pages.

Problem 1. Consider the vectors $\vec{\boldsymbol{u}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}$, $\vec{\boldsymbol{v}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{z}}$, and $\vec{\boldsymbol{w}} = \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}$.

(a) **(2 pts.)** Write the matrix $[A]$ whose columns are $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{w}}$ and compute the determinant. Specifically, let

$$[A] = \begin{pmatrix} | & | & | \\ \vec{\boldsymbol{u}} & \vec{\boldsymbol{v}} & \vec{\boldsymbol{w}} \\ | & | & | \end{pmatrix}.$$

then find $\det([A])$.

(b) **(1 pt.)** What is the volume of the parallelepiped generated by the vectors $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{w}}$?

(c) **(2 pts.)** Are the vectors $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{w}}$ linearly independent? Explain.

(d) **(2 pts.)** Explain geometrically why any inhomogeneous equation

$$[A]\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$$

(where $\vec{\boldsymbol{y}} \neq \vec{\boldsymbol{0}}$) has a unique solution. *Hint: your answer from (c) may help you.*

(e) **(2 pts.)** Without computing the eigenvalues, argue why $[A]$ cannot have an eigenvalue of zero.

(f) **(2 pts.)** Is $[A]$ a symmetric matrix? Explain.

(g) **(2 pts.)** Without computing the eigenvalues, argue why all the eigenvalues must be real.

(h) **(2 pts.)** Show that $\det([A] - [I]) = 0$ and argue why $[A]$ must have at least one eigenvalue that is equal to 1.

Problem 2. *Spin* is an observable quantum phenomenon which describes intrinsic angular momentum of particles. For example, electrons and positrons are spin-1/2 particles whereas photons are spin-1 particles. An example of a massive spin-1 particle is *triplet oxygen* in its ground state (which you can read more about here: `https://en.wikipedia.org/wiki/Triplet_oxygen`). The following matrix

$$[S]_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

is Hermitian and it describes the measurement of spin aligned along the y-axis of a *Stern-Gerlach apparatus.*

If we pass a spin-1 particle through the Stern-Gerlach apparatus then the possible observed states are the eigenvectors of $[S]_y$ and the angular momentum aligned with the y-axis is given by the respective eigenvalue.

(a) **(2 pts.)** Compute the eigenvalues of $[S]_y$.

(b) **(3 pts.)** Compute the eigenvectors of $[S]_y$.

(c) **(3 pts.)** Show that the eigenvectors are orthogonal with respect to the Hermitian inner product

$$\langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle := \sum_{i=1}^{3} u_i v_i^*,$$

where * denotes the complex conjugate.

(d) **(3 pts.)** We can prepare a spin-1 particle in the state

$$\vec{\boldsymbol{\Psi}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then, we can compute the *expected value* of the spin angular momentum of the particle $\vec{\boldsymbol{\Psi}}$ by computing

$$E([S]_y) := \langle \vec{\boldsymbol{\Psi}}, [S]_y \vec{\boldsymbol{\Psi}} \rangle.$$

Compute this expected value.