MATH 271, Exam 2

Take Home Portion

Due October 23$^{\text{rd}}$ at the start of class

Name ________________________________

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page, but no other online resources (including calculators or WolframAlpha). for this portion of the exam. **Do not discuss any problem any other person.** All of your solutions should be easily identifiable and supporting work must be shown. Ambiguous or illegible answers will not be counted as correct. **Print out this sheet and staple your solutions to it. Use a new page for each problem.**

Problem 1 ____/10 **Problem 2** ____/15

Note, these problems span two pages.

Problem 1. Sequences show up in the realm of approximation all the time. One of the first sequences one may encounter comes from *Newton's method* for calculating real roots (or zeros) to a function. The idea is that we are handed a function $f(x)$ and pick an initial guess x_0 as a candidate for a root to $f(x)$. Then, we can compute a tangent line approximation to $f(x)$ at the point x_0. That is,

$$f(x) \approx f'(x_0)(x - x_0) + f(x_0).$$

From there, we can see where this tangent line intersects the $y = 0$ axis which gives us a new x-value we call x_1. If we solve for x_1, we find that we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

From here, we can generate a recursive sequence given by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

that (typically) gives better and better approximations to a root of $f(x)$. Specifically, x_n will tend to be an better approximation to a root of $f(x)$ than x_{n-1}. This method is in fact exactly how your calculator finds zeros to a function (notice that you give bounds and an initial guess when you do this)!

Feel free to use a calculator for this problem.

(a) **(2pts.)** Consider the function $f(x) = x^2 - 1$ with the starting point of $x_0 = 2$, draw a picture that illustrates the first two iterations of Newton's method. (*Hint: This requires drawing a tangent line to $f(x)$ at two different points and telling me what the intersections of these lines with the x-axis represent.*)

(b) **(2pts.)** Write the first four terms in the sequence for Newton's method. That is, find $x_0, \ldots, x_3$ and write

$$\{x_n\}_{n=0}^{\infty} = x_0,\ x_1,\ x_2,\ x_3, \ldots.$$

(c) **(3pts.)** This sequence approaches $+\sqrt{1} = 1$ since this is the nearest root to $f(x)$ compared to our starting value. What is the first value of N where the error $|x_N - 1| < 10^{-6}$?

(d) **(3pts.)** If we instead have $g(x) = x^3 - 2x + 2$ with an initial guess of $x_0 = 0$, write the first four terms of the sequence $x_0, \ldots, x_3$ for Newton's method. Does this sequence seem to approach the real root $x \approx -1.7693$?

Problem 2. The way to define the natural logarithm is by the integral function

$$\ln(x) = \int_1^x \frac{1}{r} dr.$$

We can begin to approximate values the natural logarithm by doing the following.

(a) **(4pts.)** Let $f(r) = \frac{1}{r}$. Find the Taylor series for $f(r)$ centered around the point $a = 1$.

(b) **(4pts.)** Find the antiderivative of the series you found in (a) by integrating term by term. Just do this for the first three terms in the series.

(c) **(4pts.)** You compute the value for $\ln(2) \approx .693147$ by integrating

$$\ln(2) = \int_1^2 f(r) dr.$$

Use your result from (b) to obtain a third order approximation to $\ln(2)$. *Hint: you have a series providing the antiderivative for $f(r)$ from* (b)*; use the fundamental theorem of calculus to find the definite integral above.*

(d) **(1pts.)** How close is your approximate value to the value for $\ln(2)$ provided above? In other words, how large is the error?

(e) **(2pts.)** How would you make an improved approximation of $\ln(2)$?