MATH 271, Exam 1

Name ________________________________

Instructions No textbook, homework, calculators, phones, or smart watches may be used for this exam. The exam is designed to take 50 minutes and must be submitted at the end of the class period. All of your solutions should be easily identifiable and supporting work must be shown. You may use any part of this packet as scratch paper, but please clearly label what work you want to be considered for grading. Ambiguous or illegible answers will not be counted as correct.

Only the highest scoring five problems will be counted towards your total score. You cannot get over 75 points.

Problem 1 _____/15

Problem 2 _____/15

Problem 3 _____/15

Problem 4 _____/15

Problem 5 _____/15

Problem 6 _____/15

Total ______/75

There are extra pages between each problem for scratch work.
Please circle your answers!

Problem 1.

	T	F
(a) For any polynomial $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$ there exists n complex roots (possibly repeated).	☐	☐
(b) Euler's formula is $e^{i\theta} = \sin\theta + i\cos\theta$.	☐	☐
(c) $t = 2$ is a solution to the differential equation $2x' = tx$.	☐	☐
(d) All first order linear equations are separable.	☐	☐
(e) The concentrations A and B for the chemical reaction $$A + B \xrightarrow{k} \text{Products}$$ is modelled by the equations $\frac{dA}{dt} = -kB$ and $\frac{dB}{dt} = -kA$.	☐	☐
(f) If x_1 and x_2 are solutions to a homogeneous linear equation, then the superposition $x = x_1 + x_2$ is a solution as well.	☐	☐
(g) All second order linear equations oscillate.	☐	☐
(h) A solution $x(t)$ to an inhomogeneous second order linear equation is written as the sum $x = x_h + x_p$ where x_h solves the homogeneous equation and x_p is the particular integral.	☐	☐
(i) There are infinitely many states for the quantum particle in a 1-dimensional box.	☐	☐
(j) States for the quantum particle in the 1-dimensional box can have any energy value.	☐	☐

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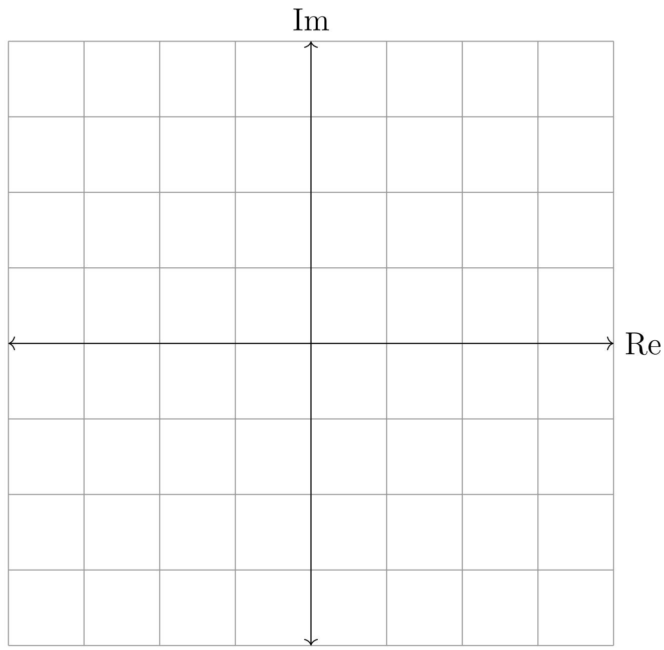
Problem 2. Let $z_1 = -1 + i$, $z_2 = 2 - i$ and $z_3 = 2e^{i\pi}$.

(a) Plot and clearly label z_1, z_2, $z_1 + z_2$, and $z_1 \cdot z_2$ on the following graph.



(b) Compute z_1^{-1} and z_3^{-1}.

(c) Write z_3 in Cartesian coordinates using Euler's Formula then plot and clearly label z_3 on the above graph.

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Problem 3. The height above ground, $y(t)$, of a ball falling through air satisfies the differential equation

$$y'' = ky' - g,$$

where k and g are positive constants.

(a) What is the order of this equation? Explain.

(b) Is this equation linear? Explain.

(c) Is this equation homogeneous or inhomogeneous? Explain.

(d) If the ball is falling through a vacuum we can let $k = 0$ so we are left with

$$y'' = -g.$$

What is the general solution to this new equation?

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Problem 4. The spring/mass harmonic oscillator is given by the equation

$$x'' + \frac{k}{m}x = 0.$$

where m is the mass of the oscillating object and k is the spring constant.

(a) Show that

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right).$$

solves the initial problem with initial data $x(0) = A$ and $x'(0) = 0$.

(b) Since this system has no damping, the total energy is <u>conserved at all times</u>. In particular, the total energy is given by

$$E = \frac{1}{2}mx'(t)^2 + \frac{1}{2}kx(t)^2 = \text{constant}.$$

and is equal for all times $t \geq 0$. What is the total energy of the given particular solution $x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right)$?

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Problem 5. For the following, explain in words how you could attempt to solve the following equations. *Hint: stating the type of differential equation is a good start!*

(a) The equation

$$x' + f(t)x = g(t).$$

(b) The equation

$$x' = f(x)g(t).$$

(c) The equation

$$x'' + bx' + cx = 0.$$

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Problem 6. Schrödinger's equation is given by

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\Psi(x) = E\Psi(x),$$

where $V(x)$ is the potential at a point x, E is the total energy, and Ψ is the wave function.

(a) For the free particle in a 1-dimensional box $[0, L]$, what are the boundary conditions for the wavefunction Ψ?

(b) Again, for the free particle in a 1-dimensional box, what is the potential inside of the box $(0, L)$?

(c) If a wavefunction Ψ is normalized, then

$$\int_0^L \|\Psi(x)\|^2 dx = 1.$$

True or false?

(d) If our wavefunction Ψ is normalized, how do we interpret the quantity

$$P([a, b]) = \int_a^b \|\Psi(x)\|^2 dx?$$

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