MATH 271, Homework 9, *Solutions*

Due November 15$^{\text{th}}$

Problem 1. Compute the following:

(a)

$$[A] = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

(b)

$$[B] = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}$$

(c) Take

$$[M] = \begin{pmatrix} 10 & 15 \\ 20 & 10 \end{pmatrix}$$

and

$$[N] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Compute $[M][N]$ and $[N][M]$ to see that matrices do not commute in general.

Solution 1.

(a) Since we have a 1×3-matrix multiplied with a 3×1-matrix, we know that $[A]$ should be a 1×1-matrix.

$$\begin{aligned} [A] &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \\ &= (1 \cdot 2 + 1 \cdot 1 + 1 \cdot 3) \\ &= (6). \end{aligned}$$

(b) Here, we should expect that $[B]$ is a 3×2-matrix.

$$[B] = \begin{pmatrix} 24 & 26 \\ 64 & 66 \\ 104 & 106 \end{pmatrix}.$$

(c) Here $[M]$ and $[N]$ are square, so multiplying will give us the same shape matrix. We have

$$[M][N] = \begin{pmatrix} 40 & 35 \\ 40 & 50 \end{pmatrix},$$

as well as

$$[N][M] = \begin{pmatrix} 50 & 35 \\ 40 & 40 \end{pmatrix}.$$

From this we can see that $[M][N] \neq [N][M]$ in general!

Problem 2. Compute the following determinants:

(a)

$$\det([A]) = \begin{vmatrix} -3 & 1 & 5 \\ -3 & 4 & 2 \\ -3 & 2 & 1 \end{vmatrix}$$

(b)

$$\det([B]) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(c) Compute $\det([A][B])$ using properties of the determinant. *Hint: this should be very quick to do. Do not compute the product of the matrices* $[A]$ *and* $[B]$*!*

Solution 2.

(a) We can expand along any row or column and in this case, there are no zeros to make the computation quicker. So we have

$$\begin{aligned} \begin{vmatrix} -3 & 1 & 5 \\ -3 & 4 & 2 \\ -3 & 2 & 1 \end{vmatrix} &= -3 \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} -3 & 2 \\ -3 & 1 \end{vmatrix} + 5 \begin{vmatrix} -3 & 4 \\ -3 & 2 \end{vmatrix} \\ &= -3(4-4) - 1(-3+6) + 5(-6+12) \\ &= 27. \end{aligned}$$

(b) Similarly, we get

$$\det([B]) = 0.$$

(c) We know that $\det([A][B]) = \det([A])\det([B])$ and thus we have that $\det([A][B]) = 0$.

Problem 3.

(a) Show that for any 2×2-matrix that the sign of the determinant changes if either a row or column is swapped. *Note: this is true for square matrices of any size.*

(b) Show that for any 2×2-matrix that multiplying a column by a constant scales the determinant by that constant as well. *Note: this is true for square matrices of any size.*

(c) Show that for any 2×2-matrix that adding a scalar multiple one column to the other will not change the determinant. *Note: this is true in broader generality. In fact, adding linear combinations of columns to another column will not change the determinant.*

Solution 3.

(a) Let

$$[A] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an arbitrary 2×2-matrix. Then we have

$$\det([A]) = ad - bc.$$

Now, if we swap rows we have

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc).$$

Now, we can do the same with columns to get

$$\begin{vmatrix} b & a \\ d & c \end{vmatrix} = bc - ad = -(ad - bc).$$

(b) Let us compute the determinant of

$$\begin{vmatrix} \alpha a & b \\ \alpha c & d \end{vmatrix} = \alpha ad - \alpha bc = \alpha(ad - bc).$$

Similarly, we will have the same if we scale the other column. In fact, this is true for rows as well.

(c) Let us add the first column to the second. We get

$$\begin{vmatrix} \alpha a & \alpha a + b \\ \alpha c & \alpha c + d \end{vmatrix} = a(\alpha c + d) - (\alpha a + b)c = \alpha ac + ad - \alpha ac - bc = ad - bc.$$

The same will be true if we add a scalar copy of column 2 to column 1.

Problem 4. * Using the facts above, argue that a square matrix with columns that are linearly dependent must have a determinant of zero.

Solution 4. Let us just show this for a 3×3-matrix as the argument is the same for the most general case. Let

$$[A] = \begin{pmatrix} | & | & | \\ \vec{\boldsymbol{A}}_1 & \vec{\boldsymbol{A}}_2 & \vec{\boldsymbol{A}}_3 \\ | & | & | \end{pmatrix}.$$

Then if the columns of $[A]$ are linearly dependent, we have that

$$\alpha_1 \vec{\boldsymbol{A}}_1 + \alpha_2 \vec{\boldsymbol{A}}_2 + \alpha_3 \vec{\boldsymbol{A}}_3 = \vec{\boldsymbol{0}}$$

with at least one $\alpha_i \neq 0$. Specifically, this means that one vector can be written as a linear combination of the others. That is we can take

$$\vec{\boldsymbol{A}}_3 = \frac{-1}{\alpha_3}(\alpha_1 \vec{\boldsymbol{A}}_1 + \alpha_2 \vec{\boldsymbol{A}}_2).$$

Then, we can subtract the quantity

$$\frac{-1}{\alpha_3}(\alpha_1 \vec{\boldsymbol{A}}_1 + \alpha_2 \vec{\boldsymbol{A}}_2)$$

from column 3 in $[A]$ to get

$$\begin{pmatrix} | & | & | \\ \vec{\boldsymbol{A}}_1 & \vec{\boldsymbol{A}}_2 & \vec{\boldsymbol{0}} \\ | & | & | \end{pmatrix},$$

which has a determinant of zero. Since we only added a linear combination of columns to another column, this did not change the determinant and hence we must have $\det([A]) = 0$.

Problem 5. What does a zero determinant indicate about the solutions of a non-homogeneous system of linear equations? (Think geometrically!)

Solution 5. If the determinant of a matrix is zero, that means the span of the columns is not the whole entire vector space. Take for example the matrix,

$$[A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The columns of this matrix only span the xy-plane. So, for example, if I take

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we get the equation

$$x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}},$$

which has no solution! Hence, a zero determinant means we cannot necessarily solve non-homogeneous equations. If we instead had the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

then this gives us the equation

$$x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}},$$

which means we must have $x = 1$ and $y = 1$. However, z is free to be anything, which means any vector of the form

$$\begin{pmatrix} 1 \\ 1 \\ z \end{pmatrix},$$

is a solution! But, this is not a unique solution.

Problem 6. What does a zero determinant indicate about the solutions of a homogeneous system of linear equations? (Think geometrically!)

Solution 6. A zero determinant for a matrix $[A]$ is equivalent to the fact that the columns are linearly dependent. Thus, this means some of the vectors are redundant. If $[A]$ is an $n \times n$-matrix, then the columns of $[A]$ could span at most $n-1$ dimensions and there must be at least one direction that is transformed to zero under the matrix $[A]$.

Take my example from the previous problem. There the columns only spanned the xy-plane and thus the z-direction is killed off by that matrix. So there were infinitely many solutions to the homogeneous equation.

Problem 7. Given the matrices

$$[A] = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -2 & -2 & 0 \end{pmatrix} \qquad \text{and} \qquad [B] = \begin{pmatrix} -3 & 1 & 1 \\ 2 & -2 & 4 \\ -1 & -1 & -1 \end{pmatrix}.$$

(a) Compute $\mathrm{tr}([A])$ and $\mathrm{tr}([B])$.

(b) Compute $\mathrm{tr}([A][B])$ and compare it to $\mathrm{tr}([B][A])$.

Solution 7.

(a) The traec is the sum of the diagonal entries. Thus we have

$$\mathrm{tr}([A]) = 1 + 1 + 0 = 2,$$
$$\mathrm{tr}([B]) = -3 - 2 - 1 = -6.$$

(b) Then we can compute $[A][B]$,

$$[A][B] = \begin{pmatrix} -5 & -1 & -1 \\ -7 & -3 & 3 \\ 2 & 2 & -10 \end{pmatrix}.$$

Hence we have

$$\mathrm{tr}([A][B]) = -18.$$

Note that under cyclic permutations, the trace is invariant, hence

$$\mathrm{tr}([A][B]) = \mathrm{tr}([B][A])$$

even though $[A][B] \neq [B][A]$

Problem 8. Consider the equation

$$[A]\vec{\boldsymbol{v}} = \vec{\mathbf{0}},$$

where

$$[A] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) What vector(s) $\vec{\boldsymbol{v}}$ satisfy this equation? In other words, what is $\text{Null}([A])$?

(b) Using what you found above, what must $\det([A])$ be equal to? *Hint: you do not need to compute the determinant!*

Solution 8. (a) To solve the homogeneous equation we take

$$[M] = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

Then we can subtract row one from row three to get

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which corresponds to the equations

$$\begin{aligned} 0x + y + 0z &= 0 \\ x + 0y + z &= 0 \\ 0x + 0y + 0z &= 0. \end{aligned}$$

Hence we have that $z = -x$ and $y = 0$. Thus any vector of the form

$$\vec{\boldsymbol{v}} = \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix}$$

for any $t \in \mathbb{R}$ is a solution to this equation. In other words, the set described above is $\text{Null}([A])$.

(b) The determinant must be equal to zero since $\text{Null}([A])$ is nontrivial (i.e., it contains more than just the zero vector).