MATH 271, Homework 9

Due November 15th

Problem 1. Compute the following:

(a)

$$[A] = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

(b)

$$[B] = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}$$

(c) Take

$$[M] = \begin{pmatrix} 10 & 15 \\ 20 & 10 \end{pmatrix}$$

and

$$[N] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Compute $[M][N]$ and $[N][M]$ to see that matrices do not commute in general.

Problem 2. Compute the following determinants:

(a)

$$\det([A]) = \begin{vmatrix} -3 & 1 & 5 \\ -3 & 4 & 2 \\ -3 & 2 & 1 \end{vmatrix}$$

(b)

$$\det([B]) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(c) Compute $\det([A][B])$ using properties of the determinant. *Hint: this should be very quick to do. Do not compute the product of the matrices $[A]$ and $[B]$!*

Problem 3.

(a) Show that for any 2×2-matrix that the sign of the determinant changes if either a row or column is swapped. *Note: this is true for square matrices of any size.*

(b) Show that for any 2×2-matrix that multiplying a column by a constant scales the determinant by that constant as well. *Note: this is true for square matrices of any size.*

(c) Show that for any 2×2-matrix that adding a scalar multiple one column to the other will not change the determinant. *Note: this is true in broader generality. In fact, adding linear combinations of columns to another column will not change the determinant.*

Problem 4. * Using the facts above, argue that a square matrix with columns that are linearly dependent must have a determinant of zero.

Problem 5. What does a zero determinant indicate about the solutions of a non-homogeneous system of linear equations? (Think geometrically!)

Problem 6. What does a zero determinant indicate about the solutions of a homogeneous system of linear equations? (Think geometrically!)

Problem 7. Given the matrices

$$[A] = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -2 & -2 & 0 \end{pmatrix} \qquad \text{and} \qquad [B] = \begin{pmatrix} -3 & 1 & 1 \\ 2 & -2 & 4 \\ -1 & -1 & -1 \end{pmatrix}.$$

(a) Compute $\mathrm{tr}([A])$ and $\mathrm{tr}([B])$.

(b) Compute $\mathrm{tr}([A][B])$ and compare it to $\mathrm{tr}([B][A])$.

Problem 8. Consider the equation

$$[A]\vec{\boldsymbol{v}} = \vec{\mathbf{0}},$$

where

$$[A] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) What vector(s) $\vec{\boldsymbol{v}}$ satisfy this equation? In other words, what is $\mathrm{Null}([A])$?

(b) Using what you found above, what must $\det([A])$ be equal to? *Hint: you do not need to compute the determinant!*