MATH 271, Homework 8, *Solutions*

Due November 8th

Problem 1. Let a mass m_1 weighing $1kg.$ be placed at $\vec{\boldsymbol{r}}_1 = 2\hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}$ and a mass m_2 of $2kg.$ be placed at $\vec{\boldsymbol{r}}_2 = 4\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}$. Where must a mass m_3 of $3kg.$ be placed so that the center of mass is at the origin $\vec{\mathbf{0}}$?

Solution 1. One can compute the center of mass $\vec{\boldsymbol{R}}_{CM}$ by

$$\vec{\boldsymbol{R}}_{CM} = \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + m_3\vec{\boldsymbol{r}}_3}{m_1 + m_2 + m_3}.$$

Here, we know everything but $\vec{\boldsymbol{r}}_3$. Since we want the center of mass at the origin $\vec{\mathbf{0}}$, then

$$\vec{\mathbf{0}} = \frac{1}{m_1 + m_2 + m_3}\left(m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + m_3\vec{\boldsymbol{r}}_3\right)$$
$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \frac{1}{6}\left(\begin{pmatrix}2\\-3\\-1\end{pmatrix} + 2\begin{pmatrix}0\\4\\-2\end{pmatrix} + 3\begin{pmatrix}x\\y\\z\end{pmatrix}\right).$$

What we have above is three equations and three unknowns. That is, one equation for the $\hat{\boldsymbol{x}}$-component, one for the $\hat{\boldsymbol{y}}$-component, and one for the $\hat{\boldsymbol{z}}$-component. We have

$$\begin{aligned}0 &= \frac{1}{6}(2 + 2\cdot 0 + 3x)\\ 0 &= \frac{1}{6}(-3 + 2\cdot 4 + 3y)\\ 0 &= \frac{1}{6}(-1 + 2\cdot(-2) + 3z).\end{aligned}$$

Taking the first, we find

$$\begin{aligned}0 &= \frac{1}{3} + \frac{1}{2}x\\ -\frac{1}{3} &= \frac{1}{2}x\\ \implies x &= -\frac{2}{3}.\end{aligned}$$

Next,

$$\begin{aligned}0 &= -\frac{1}{2} + \frac{4}{3} + \frac{1}{2}y\\ -\frac{5}{6} &= \frac{1}{2}y\\ \implies y &= -\frac{5}{3}.\end{aligned}$$

Lastly, we have

$$
\begin{aligned}
0 &= -\frac{1}{6} - \frac{2}{3} + \frac{1}{2}z \\
\frac{5}{6} &= \frac{1}{2}z \\
\implies \quad z &= \frac{5}{3}.
\end{aligned}
$$

Thus we have that $\vec{\boldsymbol{r}}_3 = -\frac{2}{3}\hat{\boldsymbol{x}} - \frac{5}{3}\hat{\boldsymbol{y}} + \frac{5}{3}\hat{\boldsymbol{z}}$.

Problem 2. Which of the following are linear transformations? For those that are not, which properties of *linearity* (the properties (i) and (ii) in our notes) fail? Show your work.

(a) $T_a \colon \mathbb{R} \to \mathbb{R}$ given by $T_a(x) = \frac{1}{x}$.

(b) $T_b \colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T_b \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(c) $T_c \colon \mathbb{R} \to \mathbb{R}^3$ given by

$$T_c(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}.$$

(d) $T_d \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T_d \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix}.$$

Solution 2.

(a) This transformation fails both properties. For (i), take

$$T_a(x+y) = \frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y} = T_a(x) + T_a(y).$$

For (ii), take

$$T_a(\alpha x) = \frac{1}{\alpha x} \neq \alpha \frac{1}{x} = \alpha T_a(x).$$

(b) This is a linear transformation. To see (i) holds, take

$$\begin{aligned} T_b(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) &= T_b \left(\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \right) \\ &= T_b \begin{pmatrix} u_x + v_x \\ u_y + v_y \\ u_z + v_z \end{pmatrix} \\ &= \begin{pmatrix} u_x + v_x \\ u_y + v_y \end{pmatrix} \\ &= \begin{pmatrix} u_x \\ u_y \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ &= T_b(\vec{\boldsymbol{u}}) + T_b(\vec{\boldsymbol{v}}). \end{aligned}$$

And for (ii), we take

$$
\begin{aligned}
T_b(\alpha\vec{\boldsymbol{v}}) &= T_b\left(\alpha\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}\right) \\
&= T_b\begin{pmatrix} \alpha v_x \\ \alpha v_y \\ \alpha v_z \end{pmatrix} \\
&= \begin{pmatrix} \alpha v_x \\ \alpha v_y \end{pmatrix} \\
&= \alpha\begin{pmatrix} v_x \\ v_y \end{pmatrix} \\
&= \alpha T_b(\vec{\boldsymbol{v}}).
\end{aligned}
$$

(c) This is not a linear transformation as both properties fail. Indeed, for (i) we take

$$
\begin{aligned}
T_c(\vec{\boldsymbol{u}}+\vec{\boldsymbol{v}}) &= T_c\left(\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}\right) \\
&= T_c\begin{pmatrix} u_x+v_x \\ u_y+v_y \\ u_z+v_z \end{pmatrix} \\
&= \begin{pmatrix} u_x+v_x \\ (u_y+v_y)^2 \\ (u_z+v_z)^3 \end{pmatrix},
\end{aligned}
$$

whereas

$$
\begin{aligned}
T_c(\vec{\boldsymbol{u}})+T_c(\vec{\boldsymbol{v}}) &= T_c\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + T_c\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \\
&= \begin{pmatrix} u_x \\ u_y^2 \\ u_z^3 \end{pmatrix} + \begin{pmatrix} v_x \\ v_y^2 \\ v_y^3 \end{pmatrix} \\
&= \begin{pmatrix} u_x+v_x \\ u_y^2+v_y^2 \\ u_z^3+v_z^3 \end{pmatrix}.
\end{aligned}
$$

Note that $u_y^2+v_y^2 \neq (u_y+v_y)^2$ and $u_z^3+v_z^3 \neq (u_z+v_z)^3$.

To see that (ii) does not hold, take

$$\begin{aligned} T_c(\alpha\vec{\boldsymbol{v}}) &= T_c \begin{pmatrix} \alpha v_x \\ \alpha v_y \\ \alpha v_z \end{pmatrix} \\ &= \begin{pmatrix} \alpha v_x \\ \alpha^2 v_y^2 \\ \alpha^3 v_z^3 \end{pmatrix}, \end{aligned}$$

whereas

$$\alpha T_c(\vec{\boldsymbol{v}}) = \begin{pmatrix} \alpha v_x \\ \alpha v_y^2 \\ \alpha v_z^3 \end{pmatrix}.$$

These are clearly not equal for every scalar α.

(d) This function is linear. For (i), we have

$$\begin{aligned} T_d(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) &= T_d \begin{pmatrix} u_x + v_x \\ u_y + v_y \end{pmatrix} \\ &= \begin{pmatrix} (u_x + v_x) + (u_y + v_y) \\ (u_x + v_x) + (u_y + v_y) \\ (u_x + v_x) + (u_y + v_y) \end{pmatrix} \\ &= \begin{pmatrix} u_x + u_y \\ u_x + u_y \\ u_x + u_y \end{pmatrix} + \begin{pmatrix} v_x + v_y \\ v_x + v_y \\ v_x + v_y \end{pmatrix} \\ &= T(\vec{\boldsymbol{u}}) + T(\vec{\boldsymbol{v}}). \end{aligned}$$

And for (ii) we have

$$\begin{aligned} T_d(\alpha\vec{\boldsymbol{v}}) &= T_d \begin{pmatrix} \alpha v_x \\ \alpha v_y \end{pmatrix} \\ &= \begin{pmatrix} \alpha v_x + \alpha v_y \\ \alpha v_x + \alpha v_y \\ \alpha v_x + \alpha v_y \end{pmatrix} \\ &= \alpha T_d(\vec{\boldsymbol{v}}). \end{aligned}$$

Problem 3. Write down the matrix for the following linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}.$$

Solution 3. We need that

$$[T]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}$$

via matrix multiplication. Since the input vector is a 3-dimensional vector, and the output vector is 3-dimensional, we must have that $[T]$ is a 3×3-matrix. Hence,

$$[T] = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t_{11}x + t_{12}y + t_{13}z \\ t_{21}x + t_{22}y + t_{23}z \\ t_{31}x + t_{32}y + t_{33}z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x \\ 3y+z \end{pmatrix}.$$

If we match the coefficients on the x, y, and z, we find that

$$[T] = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$

Problem 4. A linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

$$[T] = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

(a) Compute how T transforms the standard basis elements for $\mathbb{R}^3$. That is, find

$$T(\hat{\boldsymbol{x}}), \qquad T(\hat{\boldsymbol{y}}), \qquad T(\hat{\boldsymbol{z}}).$$

This gives a nice interpretation of matrix vector multiplication as linear combinations of the column vectors that make up a matrix.

(b) If we apply this linear transformation to the unit cube (that is, all points who have (x, y, z) coordinates with $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$), what will the volume of the transformed cube be? (*Hint: the determinant of this matrix* $[T]$ *provides us this information.*)

Solution 4.

(a) The point here is that we can understand the matrix $[T]$ and matrix multiplication better by seeing how the basis vectors are transformed. So we have

$$\begin{aligned} T(\hat{\boldsymbol{x}}) &= \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \end{aligned}$$

which is just the first column of the matrix. Then we have

$$\begin{aligned} T(\hat{\boldsymbol{y}}) &= \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \end{aligned}$$

which is just the second column of the matrix. Lastly we have

$$\begin{aligned} T(\hat{\boldsymbol{z}}) &= \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \end{aligned}$$

which is the last column of the matrix.

(b) The three basis vectors

$$\hat{\boldsymbol{x}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{y}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

define the volume of the unit cube. That is, the parallelepiped generated by $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ is the unit cube. Hence, if we know how these vectors are transformed, we just need to find the volume of the paralellepiped given by the transformed vectors $T(\hat{\boldsymbol{x}})$, $T(\hat{\boldsymbol{y}})$, and $T(\hat{\boldsymbol{z}})$. Now, we can collect these vectors into a matrix,

$$\begin{pmatrix} | & | & | \\ T(\hat{\boldsymbol{x}}) & T(\hat{\boldsymbol{y}}) & T(\hat{\boldsymbol{z}}) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},$$

which is exactly $[T]$! This is what we realized in part (a)! Now, the determinant of the matrix gives us the signed volume of the parallelepiped generated by the three column vectors, and hence

$$\text{Area} = |\det([T])| = |-7| = 7.$$

Problem 5. What does a zero determinant indjicate about the solutions of a non-homogeneous system of linear equations? (Think geometrically!)

Problem 6. What does a zero determinant indicate about the solutions of a homogeneous system of linear equations? (Think geometrically!)

Problem 7. Solve the following equation.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 11 \end{pmatrix}.$$