MATH 271, Homework 7, *Solutions*

Due November 1st

Problem 1. Let S be the set of general solutions $x(t)$ to the following homogeneous linear differential equation

$$x'' + f(t)x' + g(t)x = 0.$$

Show that this set S is a vector space over the complex numbers by doing the following. Let $x(t), y(t) \in S$ be solutions to the above equation and let $\alpha, \beta \in \mathbb{C}$ be complex scalars.

(a) Write down the eight requirements for S to be a vector space.

(b) Identify the $\vec{\mathbf{0}} \in S$ and $1 \in \mathbb{C}$.

(c) Show that $\alpha x(t) + \beta y(t) \in S$. That is, show that a superposition of solutions is also a solution. *Hint: We have shown this before.*

Solution 1. (a) We can remember these requirements via the acronym CANI ADDU. So we have for the vector addition properties

- Commutivity: If we have two solutions $x(t)$ and $y(t)$ in the set S, then we know

$$x(t) + y(t) = y(t) + x(t)$$

is satisfied.

- Associativity: If we have three solutions $x(t), y(t), z(t) \in S$, then we know

$$(x(t) + y(t)) + z(t) = x(t) + (y(t) + z(t))$$

is satisfied.

- Neutral Element: We have that there exists the zero function $0 \in S$ such that

$$0 + x(t) = x(t).$$

- Inverses: Given an $x(t) \in S$, we have the function $-x(t) \in S$ such that

$$x(t) + (-x(t)) = 0.$$

Then we have the scalar multiplication properties

- Associativity: If we have $\alpha, \beta \in \mathbb{C}$ and $x(t) \in S$ then we have

$$\alpha(\beta x(t)) = (\alpha\beta)x(t)$$

holds.

- Distribution: Given $\alpha, \beta \in \mathbb{C}$ and $x(t) \in S$ we have

$$(\alpha + \beta)x(t) = \alpha x(t) + \beta x(t)$$

holds.

- Distribution: Given $\alpha \in \mathbb{C}$ and $x(t), y(t) \in S$, we have

$$\alpha(x(t) + y(t)) = \alpha x(t) + \alpha y(t)$$

holds.

- Unit element: We have $1 \in \mathbb{C}$ satisfies that for any $x(t) \in S$ that

$$1x(t) = x(t).$$

(b) Now, note that above we defined $\vec{\mathbf{0}} \in S$ to be the zero function 0. That is, the function that is 0 for every value of t. Note that 0 is a solution to the equation since

$$0'' + f(t) \cdot 0' + g(t)0 = 0.$$

Then, we have $1 \in \mathbb{C}$ that satisfies the necessary property to. In this case, 1 is literally the unit element we care about.

Remark 1. Not all vector spaces will have such obvious neutral elements, $\vec{\mathbf{0}}$, or unit elements 1. This is why we must be a bit careful at times.

(c) Now, the biggest requirement for a vector space is that linear combinations of vectors actually produce another vector. This is quite obvious in the plane $\mathbb{R}^2$ for example, but here, it is not necessarily obvious.

Now, we take $\alpha, \beta \in \mathbb{C}$ and $x(t), y(t) \in S$ and we consider the linear combination

$$z(t) = \alpha x(t) + \beta y(t).$$

We then wish to show that this linear combination (or superposition) is a solution as well. So we plug in $z(t)$ into our equation as follows

$$\begin{aligned} z''(t) + f(t)z'(t) + g(t)z(t) &= (\alpha x''(t) + \beta y''(t)) + f(t)(\alpha x'(t) + \beta y'(t)) + g(t)(\alpha x(t) + \beta y(t)) \\ &= \alpha \left[x''(t) + f(t)x'(t) + g(t)x(t)\right] + \beta \left[y''(t) + f(t)y'(t) + g(t)y(t)\right] \\ &= 0, \end{aligned}$$

since we knew that $x(t)$ and $y(t)$ themselves are solutions. Thus, $z(t)$ is as well and now we have that S is a vector space.

Problem 2. Consider the following vectors in the real plane $\mathbb{R}^2$. We let

$$\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -3\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}}.$$

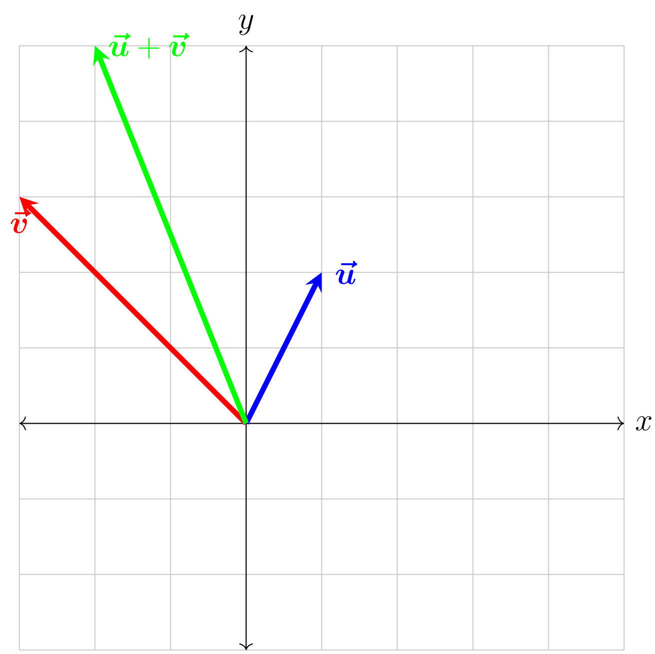
(a) Draw both $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ in the plane and label the origin.

(b) Draw the vector $\vec{\boldsymbol{w}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$ in the plane.

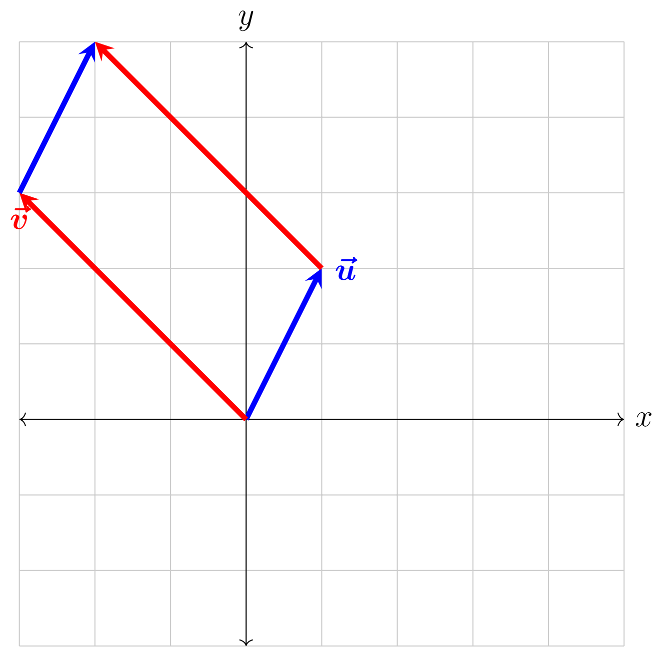
(c) Find the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

Solution 2. (a) See the plane below.

(b) Both (a) and (b) are in the plane here:



(c) One could compute the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ in many ways. First, let us see what this looks like:



In order to compute this area, we can use the cross product by thinking of these vectors as being in 3-dimensional space by

$$\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 0\hat{\boldsymbol{z}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -3\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} + 0\hat{\boldsymbol{z}}.$$

Then the cross product of these two vectors must only have a z-component since these two vectors lie in the $xy-$plane. Thus, we can compute

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = (1 \cdot 3 - (-3) \cdot 2)\hat{\boldsymbol{z}} = 9\hat{\boldsymbol{z}}.$$

Hence, the area is $\|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}\| = \|9\hat{\boldsymbol{z}}\| = 9$.

Problem 3.

(a) We can reflect a vector in the plane by first reflecting basis vectors. Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function be defined by

$$R(\hat{\boldsymbol{x}}) = -\hat{\boldsymbol{x}} \qquad \text{and} \qquad R(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{y}}.$$

Let $\vec{\boldsymbol{v}} = \alpha_1\hat{\boldsymbol{x}} + \alpha_2\hat{\boldsymbol{y}}$ and define

$$R(\vec{\boldsymbol{v}}) = \alpha_1 R(\hat{\boldsymbol{x}}) + \alpha_2 R(\hat{\boldsymbol{y}}).$$

When this is the case, we call the function T linear.
Show that R reflects the vector $\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}}$ about the y-axis and draw a picture.

(b) We can rotate a vector in the plane by first rotating the basis vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Define a linear function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}} \qquad \text{and} \qquad T(\hat{\boldsymbol{y}}) = -\hat{\boldsymbol{x}}.$$

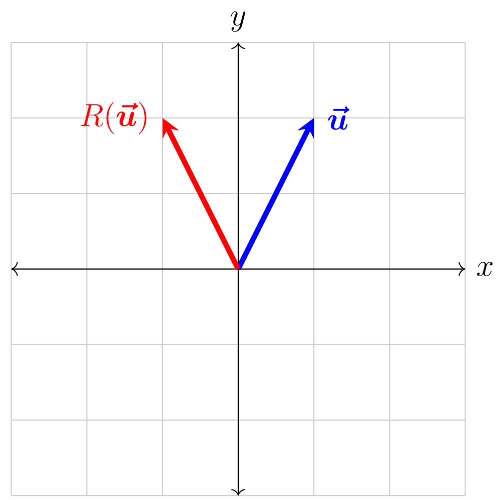
Show that T rotates $\vec{\boldsymbol{u}}$ by $\pi/2$ in the counterclockwise direction and draw a picture.

Solution 3.

(a) So, we can take the vector $\vec{\boldsymbol{u}}$ and then we have

$$R(\vec{\boldsymbol{u}}) = 1R(\hat{\boldsymbol{x}}) + 2R(\hat{\boldsymbol{y}}) = -1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}}.$$

So we can plot both $\vec{\boldsymbol{u}}$ and $R(\vec{\boldsymbol{u}})$ in the plane:

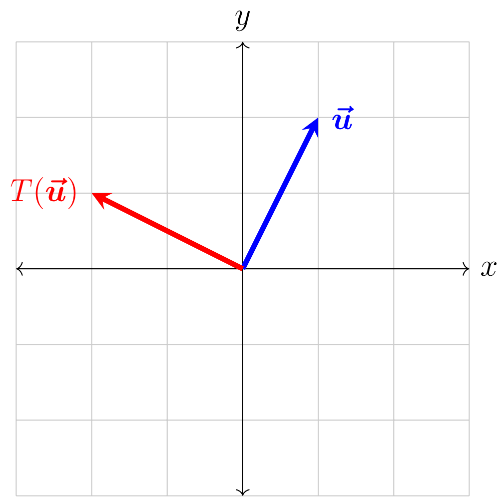


We can see that this is definitely the reflection of the vector $\vec{\boldsymbol{u}}$ across the y-axis.

(b) We can now do this for the function T to get

$$T(\vec{\boldsymbol{u}}) = 1T(\hat{\boldsymbol{x}}) + 2T(\hat{\boldsymbol{y}}) = 1\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{x}}.$$

Then we can plot both $\vec{\boldsymbol{u}}$ and $T(\vec{\boldsymbol{u}})$ in the plane:



We can see that this is definitely the rotation of the vector $\vec{\boldsymbol{u}}$ an angle of $\pi/2$ in the counter-clockwise direction.

Problem 4. Consider the following vectors in space $\mathbb{R}^3$

$$\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -2\hat{\boldsymbol{x}} + 1\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}.$$

(a) Compute the dot product $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$.

(b) Compute the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$.

(c) **(Experimental)** Let us try this: Take

$$\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} = (1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}})(-2\hat{\boldsymbol{x}} + 1\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}).$$

Distribute the above multiplication.

(d) **(Experimental)** Now, in the above multiplication, adopt the following rules:

$$\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} = 1 \qquad \hat{\boldsymbol{x}}\hat{\boldsymbol{y}} = -\hat{\boldsymbol{y}}\hat{\boldsymbol{x}} \qquad \hat{\boldsymbol{x}}\hat{\boldsymbol{z}} = -\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} \qquad \hat{\boldsymbol{y}}\hat{\boldsymbol{z}} = -\hat{\boldsymbol{z}}\hat{\boldsymbol{y}}.$$

Then, simplify the multiplication in part (c) to

$$\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} = \alpha + \beta_1\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} + \beta_2\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + \beta_3\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}.$$

That is, what are α, β_1, β_2, and β_3?

(e) **(Experimental)** If we perform one more step, we will notice something quite nice. Note that the pairs of vectors above define a plane, and there is a unique vector perpendicular to that plane. Using this fact, we can let

$$\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}} \qquad \hat{\boldsymbol{z}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \qquad \hat{\boldsymbol{x}}\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}.$$

In other words, we can replace the two vectors above with their cross product, (i.e., $\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} = \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}$.) Show that with these rules

$$\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}.$$

Solution 4.

(a) We have that

$$\begin{aligned}\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} &= 1 \cdot (-2) + 2 \cdot 1 + 3 \cdot (-3) \\ &= -6.\end{aligned}$$

(b) Here, feel free to use a formula for a cross product instead of writing it all out. We will find that

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = -7\hat{\boldsymbol{x}} - 4\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}}.$$

(c) Note, I put experimental in these following parts as they are not the traditional way of teaching these topics. However, I think this methodogology gives a more intuitive notion of vectors than the traditional dot and cross products.

So we have

$$\begin{aligned}\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} &= (1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}})(-2\hat{\boldsymbol{x}} + 1\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}) \\ &= -2\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + 1\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{x}}\hat{\boldsymbol{z}} \\ &\quad + -4\hat{\boldsymbol{y}}\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}}\hat{\boldsymbol{y}} - 4\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} \\ &\quad + -6\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{z}}\hat{\boldsymbol{y}} - 6\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}.\end{aligned}$$

(d) Now, we can use the rules to find that we get

$$\begin{aligned}\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} &= (-2 + 2 - 6) + (-4 - 3)\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} + (-6 + 2)\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + (1 + 4)\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} \\ &= -6 - 7\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} - 4\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + 5\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}.\end{aligned}$$

Hence, we have $\alpha = -6$, $\beta_1 = -7$, $\beta_2 = -4$, $\beta_3 = 5$.

Here we can imagine that there are units attached to the quantities $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$. There is a scalar quantity which contains none of these unit vectors, and there are quantities which contain two (i.e., $\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}$). The quantities containing two unit vectors represent the plane given by the two vectors.

(e) Now, we can identify a unit vector perpendicular to a plane in $\mathbb{R}^3$ that satisfies the right hand rule given by the cross product. In which case, we replace the above "multivectors" (i.e., $\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}$) with the unique unit vector perpendicular to the two. So we have

$$\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} = -6 - 7\hat{\boldsymbol{x}} - 4\hat{\boldsymbol{y}} + 5\hat{\boldsymbol{z}},$$

which is indeed giving us that

$$\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}.$$

Remark 2. This all falls under the mathematics of *geometric algebra* which is a more general way of understanding vectors. It is rather useful, yet it is not quite mainstream. Maybe one day it will be!

Problem 5. Consider the same vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^3$ from Problem 4.

(a) Compute the lengths $\|\vec{\boldsymbol{u}}\|$ and $\|\vec{\boldsymbol{v}}\|$ using the dot product.

(b) Compute the angle between vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$. *Hint: Save some work and use results from Problem 4.*

(c) Compute the projection of $\vec{\boldsymbol{u}}$ in the direction of $\vec{\boldsymbol{v}}$. *Hint: Again, save yourself some time and use results from Problem 4.*

Solution 5.

(a) Note that we have

$$\|\vec{\boldsymbol{u}}\| = \sqrt{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}}$$

and so we have

$$\|\vec{\boldsymbol{u}}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

Similarly,

$$\|\vec{\boldsymbol{v}}\| = \sqrt{(-2)^2 + 1^2 + (-2)^2} = \sqrt{9} = 3.$$

(b) Now, we found $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = -6$ and we have

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \|\vec{\boldsymbol{u}}\|\|\vec{\boldsymbol{v}}\| \cos\theta$$

as well. Hence, it follows that

$$\begin{aligned} \theta &= \arccos\left(\frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{u}}\|\|\vec{\boldsymbol{v}}\|}\right) \\ &= \arccos\left(\frac{-6}{\sqrt{14} \cdot 3}\right) \\ &\approx 2.1347 \approx 122.3^\circ. \end{aligned}$$

(c) To compute this projection of $\vec{\boldsymbol{u}}$ onto the direction of $\vec{\boldsymbol{v}}$, we must first normalize $\vec{\boldsymbol{v}}$ to make $\hat{\boldsymbol{v}}$. We have

$$\hat{\boldsymbol{v}} = \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|} = -\frac{2}{3}\hat{\boldsymbol{x}} + \frac{1}{3}\hat{\boldsymbol{y}} - \frac{2}{3}\hat{\boldsymbol{z}}.$$

Then we can compute the projection by

$$\begin{aligned} (\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}})\hat{\boldsymbol{v}} &= \left((\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}}) \cdot \left(-\frac{2}{3}\hat{\boldsymbol{x}} + \frac{1}{3}\hat{\boldsymbol{y}} - \frac{2}{3}\hat{\boldsymbol{z}}\right)\right)\hat{\boldsymbol{v}} \\ &= 2\hat{\boldsymbol{v}}. \end{aligned}$$

You could also just compute

$$\frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|^2}\vec{\boldsymbol{v}}$$

and get the same answer.