# MATH 271, Homework 7

Due November 1st

**Problem 1.** Let $S$ be the set of general solutions $x(t)$ to the following homogeneous linear differential equation

$$x'' + f(t)x' + g(t)x = 0.$$

Show that this set $S$ is a vector space over the complex numbers by doing the following. Let $x(t), y(t) \in S$ be solutions to the above equation and let $\alpha, \beta \in \mathbb{C}$ be complex scalars.

(a) Write down the eight requirements for $S$ to be a vector space.

(b) Identify the $\vec{\mathbf{0}} \in S$ and $1 \in \mathbb{C}$.

(c) Show that $\alpha x(t) + \beta y(t) \in S$. That is, show that a superposition of solutions is also a solution. *Hint: We have shown this before.*

**Problem 2.** Consider the following vectors in the real plane $\mathbb{R}^2$. We let

$$\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -3\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}}.$$

(a) Draw both $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ in the plane and label the origin.

(b) Draw the vector $\vec{\boldsymbol{w}} = \vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$ in the plane.

(c) Find the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

**Problem 3.**

(a) We can reflect a vector in the plane by first reflecting basis vectors. Let $R\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function be defined by

$$R(\hat{\boldsymbol{x}}) = -\hat{\boldsymbol{x}} \qquad \text{and} \qquad R(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{y}}.$$

Let $\vec{\boldsymbol{v}} = \alpha_1\hat{\boldsymbol{x}} + \alpha_2\hat{\boldsymbol{y}}$ and define

$$R(\vec{\boldsymbol{v}}) = \alpha_1 R(\hat{\boldsymbol{x}}) + \alpha_2 R(\hat{\boldsymbol{y}}).$$

When this is the case, we call the function $T$ <u>linear</u>.
Show that $R$ reflects the vector $\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}}$ about the $y$-axis and draw a picture.

(b) We can rotate a vector in the plane by first rotating the basis vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Define a <u>linear</u> function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}} \qquad \text{and} \qquad T(\hat{\boldsymbol{y}}) = -\hat{\boldsymbol{x}}.$$

Show that $T$ rotates $\vec{\boldsymbol{u}}$ by $\pi/2$ in the counterclockwise direction and draw a picture.

**Problem 4.** Consider the following vectors in space $\mathbb{R}^3$

$$\vec{\boldsymbol{u}} = 1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -2\hat{\boldsymbol{x}} + 1\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}.$$

(a) Compute the dot product $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$.

(b) Compute the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$.

(c) **(Experimental)** Let us try this: Take

$$\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} = (1\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + 3\hat{\boldsymbol{z}})(-2\hat{\boldsymbol{x}} + 1\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}).$$

Distribute the above multiplication.

(d) **(Experimental)** Now, in the above multiplication, adopt the following rules:

$$\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} = 1 \qquad \hat{\boldsymbol{x}}\hat{\boldsymbol{y}} = -\hat{\boldsymbol{y}}\hat{\boldsymbol{x}} \qquad \hat{\boldsymbol{x}}\hat{\boldsymbol{z}} = -\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} \qquad \hat{\boldsymbol{y}}\hat{\boldsymbol{z}} = -\hat{\boldsymbol{z}}\hat{\boldsymbol{y}}.$$

Then, simplify the multiplication in part (c) to

$$\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} = \alpha + \beta_1\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} + \beta_2\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + \beta_3\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}.$$

That is, what are $\alpha$, $\beta_1$, $\beta_2$, and $\beta_3$?

(e) **(Experimental)** If we perform one more step, we will notice something quite nice. Note that the pairs of vectors above define a plane, and there is a unique vector perpendicular to that plane. Using this fact, we can let

$$\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}} \qquad \hat{\boldsymbol{z}}\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \qquad \hat{\boldsymbol{x}}\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}.$$

In other words, we can replace the two vectors above with their cross product, (i.e., $\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} = \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}$.) Show that with these rules

$$\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}.$$

**Problem 5.** Consider the same vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^3$ from Problem 4.

(a) Compute the lengths $\|\vec{\boldsymbol{u}}\|$ and $\|\vec{\boldsymbol{v}}\|$ using the dot product.

(b) Compute the angle between vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$. *Hint: Save some work and use results from Problem 4.*

(c) Compute the projection of $\vec{\boldsymbol{u}}$ in the direction of $\vec{\boldsymbol{v}}$. *Hint: Again, save yourself some time and use results from Problem 4.*