MATH 271, Homework 6, *Solutions*

Due October 18^{th}

Problem 1. Consider the differential equation

$$f'(x) = \frac{1}{\sqrt{1-x^2}} f(x).$$

(a) Write down the 2^{nd} order Taylor approximation to $\frac{1}{\sqrt{1-x^2}}$ centered at zero.

(b) Using this second order approximation, find the general solution to the differential equation using separation.

(c) The solution you find using the approximation doesn't have an issue at $x = 1$, but I claim the original equation does. What is wrong at $x = 1$? Our approximation is then only reasonable in the window $[0, 1)$ (and really isn't that accurate near 1 either).

Solution 1.

(a) We need only compute up to the second derivative of $\frac{1}{\sqrt{1-x^2}}$ to get the desired approximation. So we have

$$\begin{aligned} f^{(0)}(x) &= \frac{1}{\sqrt{1-x^2}} && \implies f^{(0)}(0) = 1 \\ f^{(1)}(x) &= \frac{x}{(1-x^2)^{3/2}} && \implies f^{(1)}(0) = 0 \\ f^{(2)}(x) &= \frac{3x^2}{(1-x^2)^{5/2}} + \frac{1}{(1-x^2)^{3/2}} && \implies f^{(2)}(0) = 1. \end{aligned}$$

Hence we have that

$$\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{x^2}{2}$$

to second order.

(b) Now the approximate equation is

$$f'(x) \approx \left(1 + \frac{x^2}{2}\right) f(x)$$

which we can solve using separation. Thus,

$$\begin{aligned} \frac{1}{f} df &= \left(1 + \frac{x^2}{2}\right) dx \\ \ln(f) &= x + \frac{x^3}{6} + C \\ \implies f &= Ce^{x + \frac{x^3}{6}}. \end{aligned}$$

(c) As $x \to 1^-$ in the above equation, we have that the right hand side may approach infinity since

$$\lim_{x \to 1^-} \frac{1}{1 - x^2} = \infty.$$

Now, if $f(x) \to 0$ quickly enough, it could be that these effects mitigate each other to some extent, but this is not the case. We have that if $f(0) = 0$, then the solution is stationary. If $f(0) > 0$ the solution will grow to infinity by the point $x = 1$ since $f(x)$ and $f'(x)$ will both be positive we already showed the above limit. Similarly, if $f(0) < 0$, then the solution grows to negative infinity by the point $x = -1$ for analogous reasons.

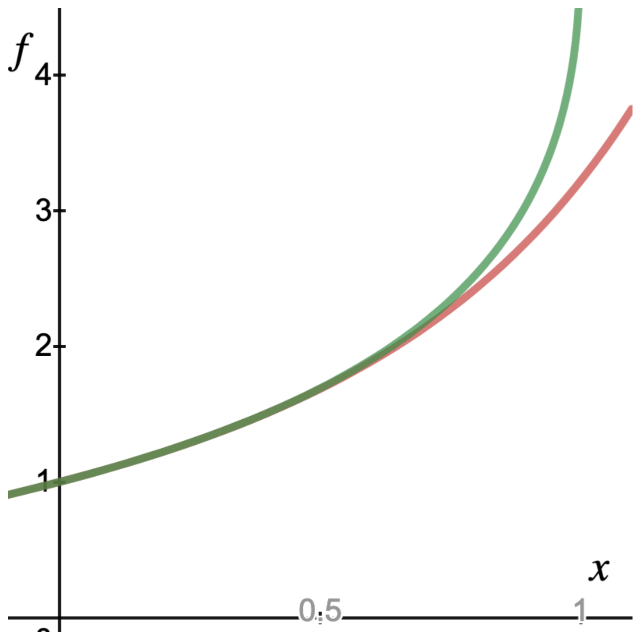

Figure 1: A graph of the true solution (green), and approximate solution (red).

Problem 2. Consider the differential equation

$$f'(x) = xf(x)$$

with initial condition $f(0) = 1$.

(a) Find the particular solution to this differential equation using separation.

(b) What is the Taylor series centered at zero for this solution?

(c) Now, assume that the solution $f(x)$ can be written as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Determine all of the coefficients a_n which will give us the power series representation for $f(x)$. *Hint: use your solution from (a) to help you.*

Solution 2.

(a) Using separation,

$$\begin{aligned} f' &= xf \\ \frac{1}{f} df &= x dx \\ \ln(f) &= \frac{x^2}{2} + C \\ f &= Ae^{\frac{x^2}{2}}. \end{aligned}$$

Then with $f(0) = 1$, we have

$$1 = A,$$

so the particular solution is

$$f(x) = e^{\frac{x^2}{2}}.$$

(b) Note that we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and thus

$$e^{\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}.$$

Now, to solve this using a power series, we assume the ansatz that $f(x)$ takes the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then we also have

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and we can plug both series into the original equation to get

$$\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} &= x \sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} a_n x^{n+1}.
\end{aligned}$$

So we can solve for the coefficients a_n to determine $f(x)$,

$$\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} a_n x^{n+1} \\
\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n &= 0 \\
a_1 + \sum_{n=1}^{\infty} \left[(n+1) a_{n+1} - a_{n-1}\right] x^n &= 0.
\end{aligned}$$

Hence we must have that $a_1 = 0$ and that

$$(n+1) a_{n+1} - a_{n-1} = 0,$$

which means that

$$a_{n+1} = \frac{1}{n+1} a_{n-1}.$$

Since $a_1 = 0$, we have that all odd terms $a_{2n+1} = 0$ by the above relationship. Then we have for the even terms

$$\begin{aligned}
a_2 &= \frac{1}{2} a_0 = \frac{1}{2^1} \cdot \frac{1}{1!} a_0 \\
a_4 &= \frac{1}{4} a_2 = \frac{1}{4} \cdot \frac{1}{2} a_0 = \frac{1}{2^2} \cdot \frac{1}{2!} a_0 \\
a_6 &= \frac{1}{6} a_4 = \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} a_0 = \frac{1}{2^3} \cdot \frac{1}{3!} a_0 \\
&\vdots \\
\implies \quad a_{2n} &= \frac{1}{2^n} \cdot \frac{1}{n!} a_0.
\end{aligned}$$

Hence our general solution is

$$f(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}.$$

If we require that $f(0) = 1$, then $a_0 = 1$ and we have

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!},$$

which is exactly what we found in (b).

Problem 3. Consider the differential equation

$$(x-1)f'(x) + f(x) = 0$$

with initial condition $f(0) = 1$.

(a) Find the solution to this equation using separation.

(b) Find the Taylor series centered at zero for your solution in (a).

(c) Again, suppose that the solution can be written as a power series and determine all the coefficients a_n so that we find the power series representation for $f(x)$. *Hint: use your solution from (a) to help you.*

Solution 3.

(a)

Problem 4. We derived two linearly independent (even and odd) solutions to *Legendre's equation*

$$(1 - x^2)f''(x) - 2xf'(x) + l(l + 1)f(x) = 0$$

which were

$$f(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} \qquad \text{and} \qquad f(x) = \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}.$$

(a) Look up where this equation shows up in quantum mechanics and write it down.

(b) If we add initial conditions then we get a finite polynomial for each choice of $\alpha = 0, 1, 2, 3, \ldots$. Using this, the first four polynomials are

$$f_0(x) = 1 \qquad f_1(x) = x$$
$$f_2(x) = 1 - 3x^2 \qquad f_3(x) = x - \frac{5x^3}{3}.$$

Show that these above polynomials are *orthogonal* by showing

$$\int_{-1}^{1} f_i(x)f_j(x)dx = 0$$

when $i \neq j$.

Solution 4.

(a) This equation arises in quantum mechanics when solving for the solution to the Hydrogen atom. Specifically, one finds the differential equation

$$\frac{d^2y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + \left[(l(l + 1) - \frac{m^2}{\sin^2\theta}\right] y = 0.$$

If we take $m = 0$ in the above equation, then we arrive at the Legendre equation provided above, but in the variable $x = \cos\theta$. This variable represents the polar angle part of the solution found using separation of variables for the central Coulomb potential for a proton and electron (i.e., the Hydrogen atom).

(b) We simply have to compute integrals for the following pairs of (i, j):

$$(0, 1) \quad (0, 2) \quad (0, 3) \quad (1, 2) \quad (1, 3) \quad (2, 3).$$

We compute each

$$\int_{-1}^{1} f_0(x)f_1(x)dx = \int_{-1}^{1} 1 \cdot xdx$$
$$= 0,$$

since x is an odd function on a symmetric interval about $x = 0$.

Next, we take

$$\begin{aligned}\int_{-1}^{1} f_0(x)f_2(x)dx &= \int_{-1}^{1} 1\cdot(1-3x^2)dx \\ &= \int_{-1}^{1} dx - 3\int_{-1}^{1} x^2dx \\ &= 2 - \left[x^3\right]_{-1}^{1} \\ &= 0.\end{aligned}$$

Next,

$$\begin{aligned}\int_{-1}^{1} f_0(x)f_3(x)dx &= \int_{-1}^{1} 1\cdot\left(x-\frac{5x^3}{3}\right)dx \\ &= 0,\end{aligned}$$

since $f_3(x)$ is an odd function.

Next,

$$\begin{aligned}\int_{-1}^{1} f_1(x)f_2(x)dx &= \int_{-1}^{1} x\cdot\left(1-3x^2\right)dx \\ &= 0,\end{aligned}$$

since $f_1(x)$ is an odd function and $f_2(x)$ is an even function and an even function times an odd function is an odd function.

Next,

$$\begin{aligned}\int_{-1}^{1} f_1(x)f_3(x)dx &= \int_{-1}^{1} x\cdot\left(x-\frac{5x^3}{3}\right)dx \\ &= \int_{-1}^{1} x^2dx - \frac{5}{3}\int_{-1}^{1} x^4dx \\ &= \frac{2}{3} - \frac{5}{3}\cdot\frac{2}{5} \\ &= 0.\end{aligned}$$

Lastly, we take

$$\begin{aligned}\int_{-1}^{1} f_2(x)f_3(x)dx &= \int_{-1}^{1} \left(1-3x^3\right)\cdot\left(x-\frac{5x^3}{3}\right) \\ &= 0,\end{aligned}$$

again since the product of an even and odd function is odd. Hence, we have shown the orthogonality relationship between all the relevant functions.