MATH 271, Homework 6

Due October 18^{th}

Problem 1. Consider the differential equation

$$f'(x) = \frac{1}{\sqrt{1-x^2}} f(x).$$

(a) Write down the 2^{nd} order Taylor approximation to $\frac{1}{\sqrt{1-x^2}}$ centered at zero.

(b) Using this second order approximation, find the general solution to the differential equation using separation.

(c) The solution you find using the approximation doesn't have an issue at $x = 1$, but I claim the original equation does. What is wrong at $x = 1$? Our approximation is then only reasonable in the window $[0, 1)$ (and really isn't that accurate near 1 either).

Problem 2. Consider the differential equation

$$f'(x) = xf(x)$$

with initial condition $f(0) = 1$.

(a) Find the particular solution to this differential equation using separation.

(b) What is the Taylor series centered at zero for this solution?

(c) Now, assume that the solution $f(x)$ can be written as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Determine all of the coefficients a_n which will give us the power series representation for $f(x)$. *Hint: use your solution from (a) to help you.*

Problem 3. Consider the differential equation

$$(x-1)f'(x) + f(x) = 0$$

with initial condition $f(0) = 1$.

(a) Find the solution to this equation using separation.

(b) Find the Taylor series centered at zero for your solution in (a).

(c) Again, suppose that the solution can be written as a power series and determine all the coefficients a_n so that we find the power series representation for $f(x)$. *Hint: use your solution from (a) to help you.*

Problem 4. We derived two linearly independent (even and odd) solutions to *Legendre's equation*

$$(1-x^2)f''(x) - 2xf'(x) + \alpha(\alpha+1)f(x) = 0$$

which were

$$f(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} \qquad \text{and} \qquad f(x) = \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}.$$

(a) Look up where this equation shows up in quantum mechanics and write it down.

(b) If we add initial conditions then we get a finite polynomial for each choice of $\alpha = 0, 1, 2, 3, \ldots$. Using this, the first four polynomials are

$$f_0(x) = 1 \qquad\qquad f_1(x) = x$$

$$f_2(x) = 1 - 3x^2 \qquad\qquad f_3(x) = x - \frac{5x^3}{3}.$$

Show that these above polynomials are *orthogonal* by showing

$$\int_{-1}^{1} f_i(x)f_j(x)dx = 0$$

when $i \neq j$.