MATH 271, Homework 4

Due October 4^{th}

Problem 1. Consider the following sequences,

$$a_n = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots,$$

and

$$b_n = 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \ldots, \frac{1}{n!}, \ldots.$$

(a) For what values of N do we need for $a_N < 0.01$ and $b_N < 0.01$? Note, these will be different values for N.

(b) Compute $\lim_{n\to\infty} a_n$.

(c) Compute $\lim_{n\to\infty} b_n$.

(d) Which sequence converges more quickly to its limit? (*Hint: consider a ratio of the terms of the sequences and take a limit. Part (a) should help you think about this.*)

Problem 2. With the same a_n from 1, consider the series

$$A = \sum_{n=1}^{\infty} a_n.$$

(a) Write down the N^{th} partial sum A_N for this series.

(b) Does this sequence of partial sums converge? If so, to what?

(c) Note that this is an *geometric series* with $a = 1$ and $r = \frac{1}{2}$. However, we start from $n = 1$ instead of $n = 0$. Show the value that this series converges to using the formula for a geometric series.

Problem 3. With the same b_n from 1, consider the series

$$B = \sum_{n=0}^{\infty} b_n.$$

(a) Use the ratio test to show that this series converges.

(b) Approximate the value the series converges to by considering larger and larger partial sums.

(c) What number does this series converge to?

Problem 4. Consider the two series

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{and} \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

(a) Show that $\cos(-x) = \cos(x)$.

(b) Show that $\sin(-x) = -\sin(x)$.

(c) To take a derivative of a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ we can do the following:

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \frac{d}{dx} x^n.$$

Compute $\frac{d}{dx} \sin(x)$ and $\frac{d}{dx} \cos(x)$ and show that they are equal to what you already know. *Warning: be careful with the powers of x in the case with* $\sin$ *and* $\cos$*!*

Problem 5. Consider the *p-series*:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

(a) For $p = 1$, show that the ratio test is inconclusive.

(b) For $p = 2$, show that the ratio test is again inconclusive.

(c) Look up the sum of the series for $p = 1$ and $p = 2$. Notice how the ratio test is not perfect!