MATH 271, Homework 3, *Solutions*

Due September 20^{TH}

Problem 1. Consider the second order chemical reaction given by

$$A + B \xrightarrow{k} \text{Products.}$$

(a) Write a *system* of differential equations to describe the concentration of the reactants A and B (this means write one for each).

(b) The concentrations of A and B can be related to each other in the following way: Let $A = A_0 - x$ and $B = B_0 - x$. Here, we think of x as the amount of each chemical that has reacted, and note that it depends on time t. Use this change of variables to rewrite the differential equation for chemical A in terms of x and t.

(c) Solve the differential equation in (b) with the initial condition $x(0) = 0$.You will need to use *partial fraction decomposition* to evaluate the integral.

Solution 1.

(a) The system of equations we will get is

$$-\frac{dA}{dt} = kAB$$
$$-\frac{dB}{dt} = kAB.$$

(b) Now, let $A = A_0 - x$ and $B = A_0 - x$ and, since A_0 and B_0 are constant, we get the equation for A,

$$-\frac{dx}{dt} = k(A_0 - x)(B_0 - x).$$

It turns out B has the same equation.

(c) This is a separable equation, so we can find the solution by

$$-\frac{dx}{dt} = k(A_0 - x)(B_0 - x)$$
$$\int \frac{dx}{(A_0 - x)(B_0 - x)} = -k \int dt.$$

Here, we can use the partial fraction decomposition to get

$$\frac{1}{A_0 - B_0} \log \left(\frac{x - A_0}{x - B_0} \right) = -kt + C.$$

Then we can find

$$\frac{x - A_0}{x - B_0} = e^{-kt+C}$$

With $x(0) = 0$ we have

$$\frac{-A_0}{-B_0} = e^{-kt}e^C$$

and so $e^C = \frac{A_0}{B_0}$. We can rewrite this in terms of A and B as

$$\frac{A}{B} = \frac{A_0}{B_0}e^{-kt}.$$

Problem 2. If $x_1(t)$ and $x_2(t)$ are solutions to the differential equation

$$x'' + bx' + cx = 0$$

is $x = x_1 + x_2 + k$ for a constant k always a solution? Is the function $y = tx_1$ a solution?

Solution 2. x and y are *not* solutions. Let's see why. We note that x_1 and x_2 are solutions and thus

$$x_i'' + bx_i' + cx_i = 0 \qquad \text{for } i = 1, 2.$$

Now, we check if x is a solution by plugging into the left hand side

$$\begin{aligned} x'' + bx' + cx &= (x_1 + x_2 + k)'' + b(x_1 + x_2 + k)' + c(x_1 + x_2 + k) \\ &= \underbrace{x_1'' + bx_1' + cx_1}_{=0} + \underbrace{x_2'' + bx_2' + cx_2}_{=0} + ck \\ &= ck \neq 0. \end{aligned}$$

So this x is not a solution.

Similarly, we take $y = tx_1$ and plug it into the left hand side and find

$$\begin{aligned} y'' + by' + cy &= (tx_1)'' + b(tx_1)' + c(tx_1) \\ &= tx_1'' + 2x_1' + b(tx_1' + x_1) + c(tx_1) \\ &= t\underbrace{(x_1'' + bx_1' + cx_1)}_{=0} + 2x_1' + bx_1 \\ &= 2x_1' + bx_1, \end{aligned}$$

which is not in general a solution unless $x_1 = 0$.

Problem 3. Consider the following initial value problem:

$$x'' + 4x' + 3x = 0$$

with initial data $x(0) = 1$, $x'(0) = 0$.

(a) Find the solution.

(b) Sketch a plot of the solution.

(c) Explain in words what is happening to the solution as time goes on. What happens as $t \to \infty$?

Solution 3.

(a) We can solve this homogeneous second order linear equation with constant coefficients by finding roots to its characteristic polynomial. In this case, that amounts to

$$\begin{aligned} &\lambda^2 + 4\lambda + 3 = 0 \\ \iff &(\lambda + 3)(\lambda + 1) = 0, \end{aligned}$$

so the roots are $\lambda_1 = -1$ and $\lambda_2 = -3$. Thus our general solution is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-t} + C_2 e^{-3t}.$$

Then we use the initial conditions to find a particular solution. Namely,

$$\begin{aligned} 1 = x(0) &= C_1 e^{-0} + C_2 e^{-3\cdot 0} = C_1 + C_2 \\ 0 = x'(0) &= -C_1 e^{-0} - 3C_2 e^{-3\cdot 0} = -C_1 - 3C_2. \end{aligned}$$

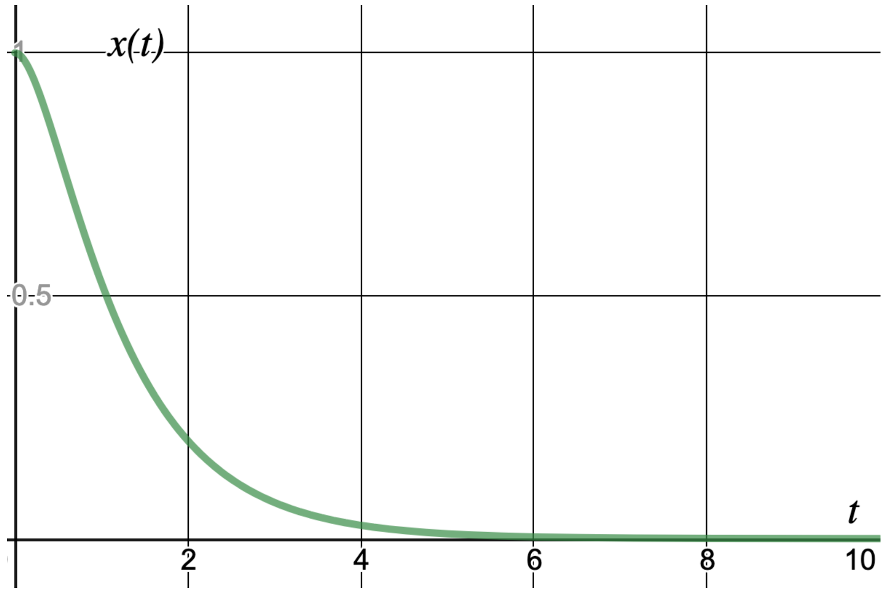
Using the second equation we get $C_1 = -3C_2$. We can plug this into the first equation to get

$$1 = -3C_2 + C_2 = -2C_2$$

meaning that $C_2 = -\frac{1}{2}$. Thus $C_2 = \frac{3}{2}$. Hence, our particular solution for this IVP is

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$

(b) Here is a plot of the particular solution from time $t = 0$ to time $t = 10$.



(c) The solution decays exponentially over time. As $t \to \infty$ our solution approaches zero.

Problem 4. Write down a homogeneous second-order linear differential equation where the system displays a decaying oscillation.

Solution 4. Since our solution should oscillate and decay, we need some form of a "spring" and some form of damping. These terms show up respectively as b and c in the equation

$$x'' + bx' + cx = 0.$$

Now, also note that (aside from one special case of two of the same real roots), our general solution has the form

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

where λ_1 and λ_2 are roots to the characteristic polynomial

$$\lambda^2 + b\lambda + c = 0.$$

Now, the roots for the characteristic polynomial are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

- To have oscillation, our roots must have an imaginary part and thus

  $$b^2 - 4c < 0.$$

  In other words, $b^2 < 4c$.

- To have a decaying solution, the real part of the roots must be negative. The real part of the roots will be $\frac{-b}{2}$ and thus we need

  $$\frac{-b}{2} < 0.$$

Now, I'll choose $b = 1$ and $c = 1$ which satisfy both of these requirements. We then have

$$x'' + x' + x = 0$$

as our equation.

Note, we can also find the solution as the roots are then

$$\lambda = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}.$$

Plugging this into the form for the general solution and we get

$$x(t) = e^{-\frac{1}{2}t} \left(C_1 \sin\left(\frac{\sqrt{3}}{2}\right) + \cos\left(\frac{\sqrt{3}}{2}\right)\right)$$

Problem 5. Consider the following differential equation:

$$x'' + 2x' + x = 3e^{-t} + 2t.$$

(a) Find the homogeneous solution $x_H(t)$.

(b) Find the particular integral $x_P(t)$.

(c) Find the specific solution corresponding to the initial data $x(0) = 0$, $x'(0) = 0$.

Solution 5.

(a) The roots to characteristic polynomial satisfy

$$\lambda^2 + 2\lambda + 1 = 0$$

which can be found by factoring

$$(\lambda + 1)^2 = 0,$$

which gives us that $\lambda = -1$ is the only root. Thus, this is the special case where our general solution looks slightly different. We'll have

$$x_h(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

(b) The right hand side has a e^{-t} term which is already present in our x_h. In fact, this means we have to take kt^2e^{-t} as a guess for this part of x_p. Then, we also have a $2t$ term, so our x_p should be

$$x_p = kt^2e^{-t} + a_0 + a_1 t.$$

Now we have to find the undetermined coefficients by plugging in and solving

$$\begin{aligned} x_p'' + 2x_p' + x_p &= 3e^{-t} + 2t \\ 2ke^{-t} - 4kte^{-t} + kt^2e^{-t} + 2(2kte^{-t} - kt^2e^{-t} + a_1) + kt^2e^{-t}a_1 t + a_0 &= 3e^{-t} + 2t \end{aligned}$$

which gives us that $k = \frac{3}{2}$, $a_1 = 2$, and $a_0 = -4$. So

$$x_p(t) = \frac{3}{2}t^2e^{-t} + 2t - 4.$$

(c) Now, we take it that our solution is of the form

$$x(t) = x_h + x_p = C_1 e^{-t} + C_2 t e^{-t} + \frac{3}{2}t^2e^{-t} + 2t - 4.$$

If we take

$$0 = x(0) = C_1 - 4$$

then $C_1 = 4$, and

$$0 = x'(0) = -4 + C_2 + 2$$

so $C_2 = 2$. Thus, our specific solution is

$$x(t) = (4 + 2t)e^{-t} + \frac{3}{2}t^2e^{-t} + 2t - 4.$$