MATH 271, HOMEWORK 2, *Solutions*

DUE SEPTEMBER 13TH

Problem 1. Solve the following autonomous equation.

$$x' = -x^2.$$

Can $x(0) = 0$ be an initial condition?

Solution 1. We have seen this equation arise from studying chemical reactions. Specifically, this equation could model

$$2x \xrightarrow{k=1} \text{Products}.$$

This equation is also separable. Which means we can solve by

$$x' = \frac{dx}{dt} = -x^2$$
$$\iff -\frac{dx}{x^2} = dt.$$

We can then integrate both sides to find

$$\frac{1}{x} = t + C.$$

Then we solve for x to find our general solution

$$\boxed{x = \frac{1}{t+c}.}$$

Since this equation could model the above reaction, we should expect that the initial condition of $x(0) = 0$ works. Why is that? If we start with no reactants (i.e., this initial condition), then no reaction should occur! We can see that by noting

$$x'(0) = -x(0)^2 = 0$$

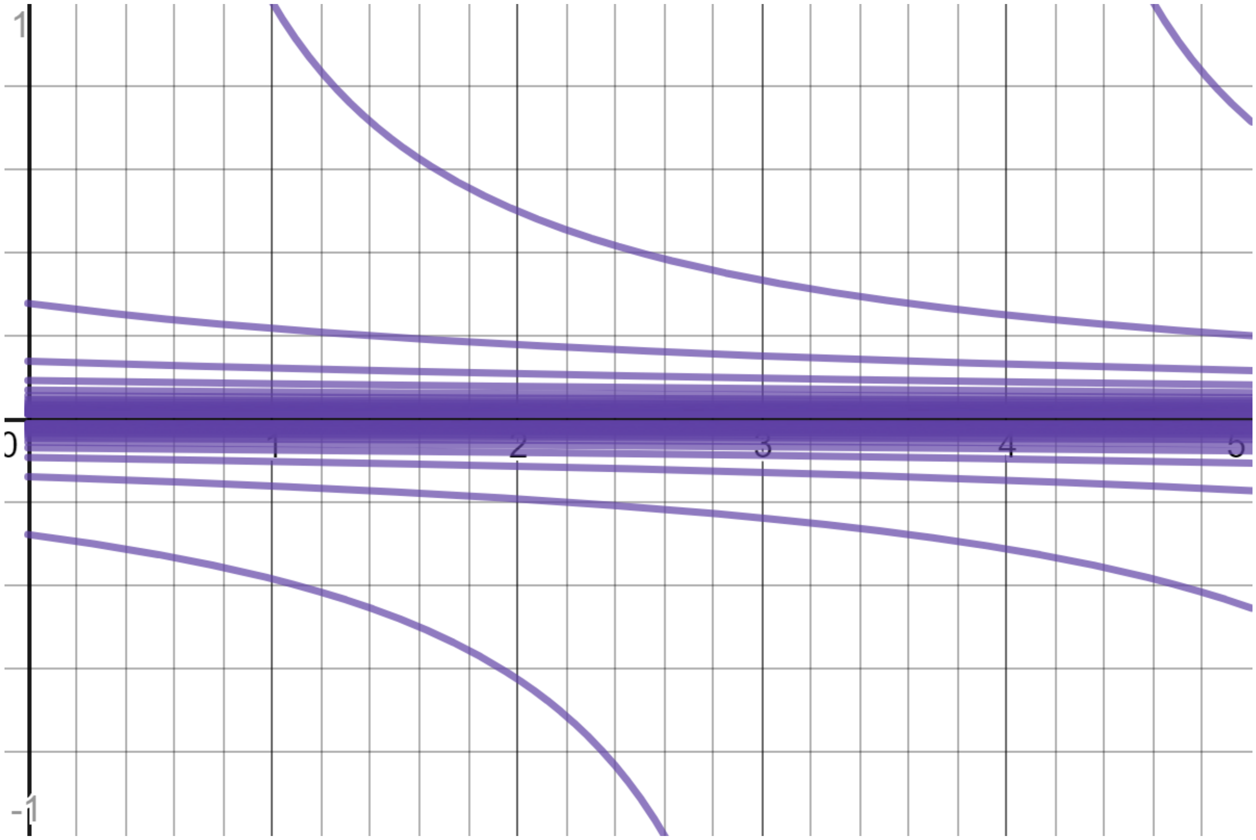
which shows that $x'(0) = 0$ given this initial condition. In that case the solution is trivial and no dynamics occur.

However, just by looking at solving the equation we have from our general solution

$$x(0) = \frac{1}{0+c} = \frac{1}{c}$$

it is not really possible to solve this equation. However, if we consider something like the $\lim_{c\to\infty} \frac{1}{t+c} = 0$ then this may make some sense. Below is a graph of many different values of c. Notice that they approach the case $x(t) = 0$ as a solution as $t \to \infty$. So in the case that $x(0) = 0$ the particular solution would be

$$\boxed{x(t) = 0.}$$


1
-1
1
2
3
4
5

Problem 2. Objects near Earth fall due to gravity. The acceleration of an object due to gravity is then

$$x'' = g,$$

where x represents the distance above the ground and $g \approx -9.8\frac{m}{s^2}$.

(a) Find the general solution to the equation.

(b) Given the initial data $x(0) = 0$ and $x'(0) = 1$, find the particular solution.

(c) Plot your solution over a meaningful range of time.

(d) When is the object touching the ground?

Solution 2.

(a) This equation can be solved by integrating twice. However, I like to make a substitution of $y = x'$ so that we can write

$$y' = g.$$

Now, this is a first order separable equation which we can solve by

$$\begin{aligned} \frac{dy}{dt} &= g \\ \int dy &= \int g dt \\ y &= gt + C_1. \end{aligned}$$

Now, since $y = x'$ we can look at

$$x' = gt + C_1$$

which is also separable. We integrate and find

$$\begin{aligned} \frac{dx}{dt} &= gt + C_1 \\ \int dx &= \int gt + C_1 dt \\ x(t) &= \frac{1}{2}gt^2 + C_1 t + C_2 \end{aligned}$$

is our general solution.

(b) Now, we use our initial data along with our general solution

$$\begin{aligned} 0 = x(0) &= \frac{1}{2}g(0)^2 + C_1(0) + C_2 \\ &= C_2. \end{aligned}$$

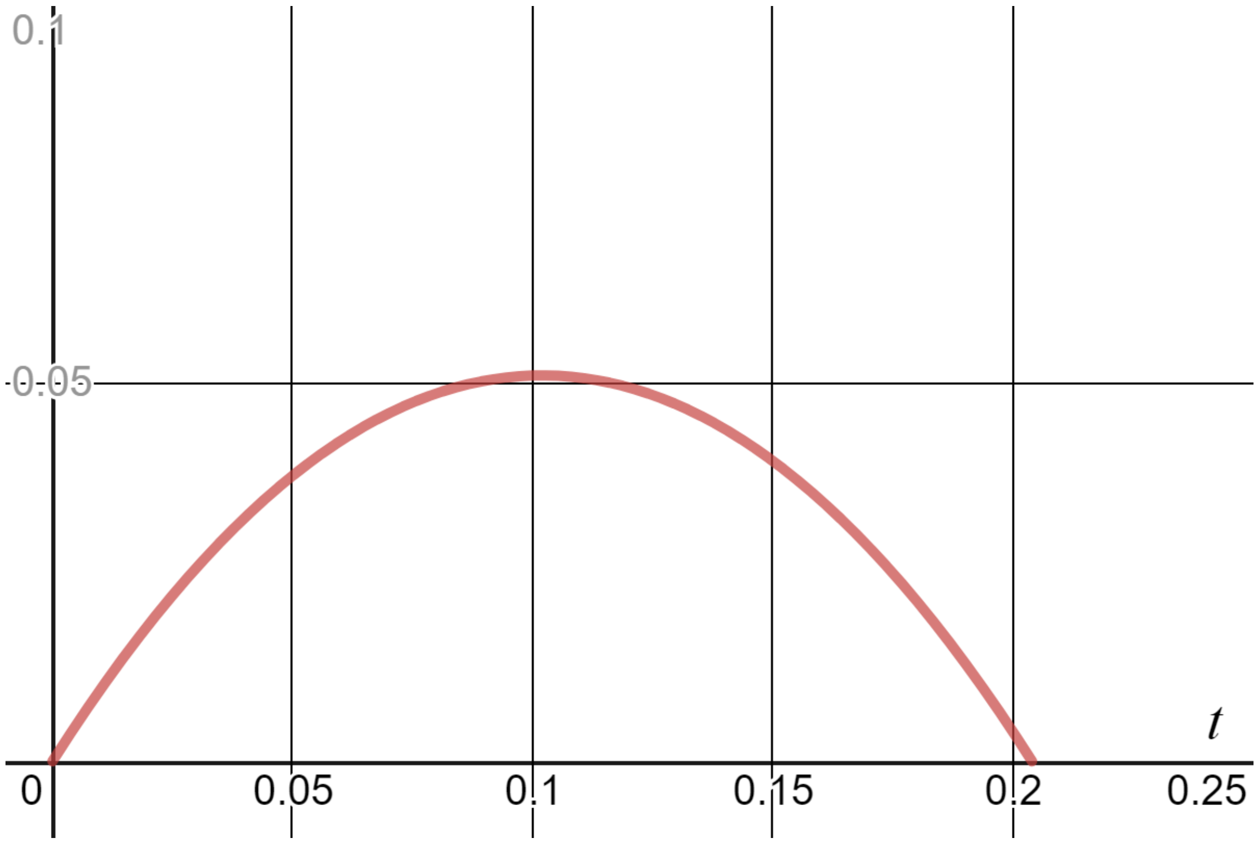
So $C_2 = 0$. Next, use the information about the first derivative $x'(0) = 1$ and we have

$$\begin{aligned} 1 = x'(0) &= g(0) + C_1 \\ &= C_1. \end{aligned}$$

Thus $C_1 = 1$. Now, our particular solution is

$$\boxed{x(t) = \frac{1}{2}gt^2 + t.}$$

(c) Here's a plot of the solution from time $t = 0$ until it hits the ground (see (d)).



(d) This was maybe better to think about before plotting in (c)! Now, the points at which the object is on the ground are the values of t where $x(t) = 0$ since x represents the height above the ground. So we have to solve

$$0 = x(t) = \frac{1}{2}gt^2 + t = t\left(\frac{1}{2}gt + 1\right).$$

This has roots $t = 0$ and $t = -\frac{2}{g}$

Problem 3. Consider the following differential equation.

$$x' = x\cos(t).$$

(a) What is the order of this equation?

(b) Find the general solution to this equation.

(c) Given the initial data $x(0) = 1$, find the particular solution.

(d) Plot this function and explain in words what the solution represents if $x(t)$ is position.

Solution 3.

(a) This is a first order equation that is also separable.

(b) We can find the general solution by

$$\begin{aligned} \frac{dx}{dt} &= x\cos(t) \\ \int \frac{1}{x}dx &= \int \cos(t)dt \\ \ln(x) &= \sin(t) + C. \end{aligned}$$

Then we solve for x to find

$$\begin{aligned} x &= e^{\sin(t)+C} = e^C \cdot e^{\sin(t)} \\ &= Ae^{\sin(t)}, \end{aligned}$$

which is our general solution.

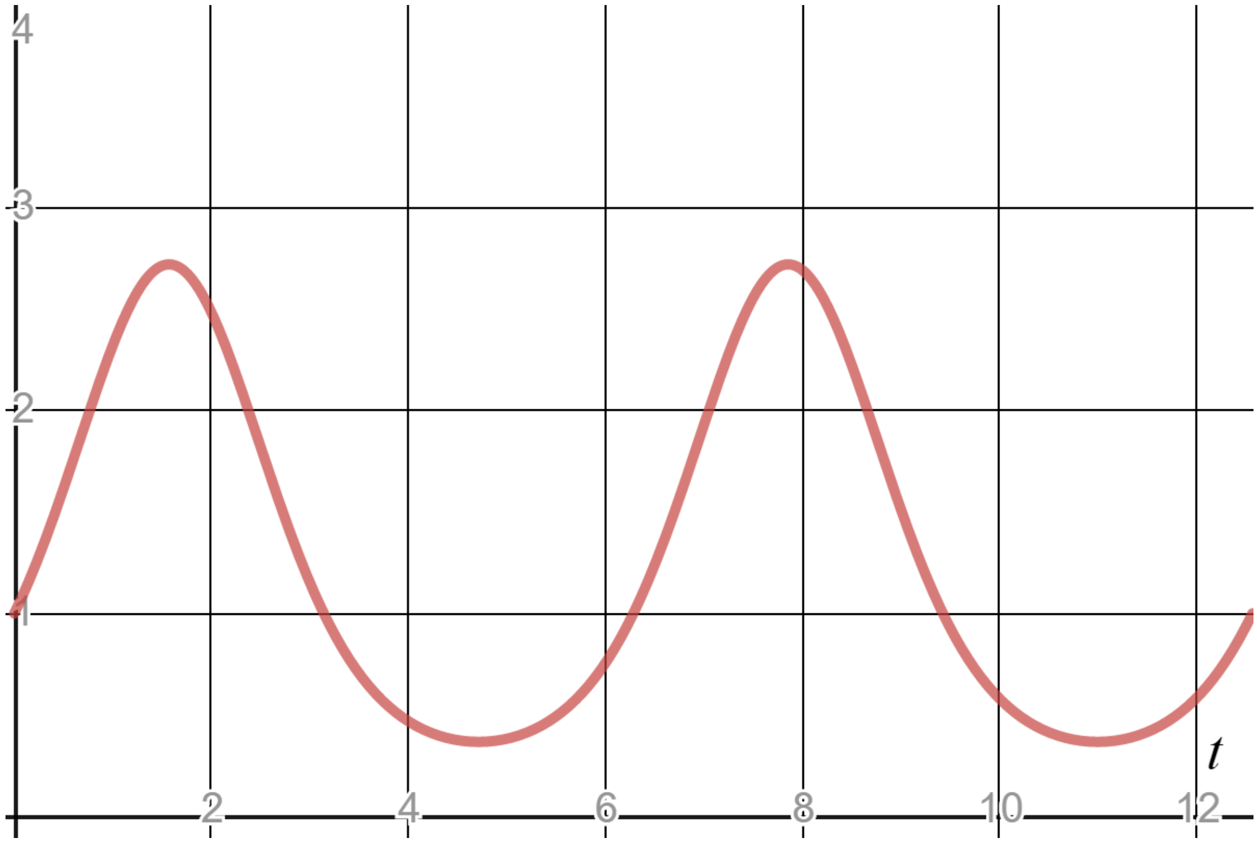
(c) If we have $x(0) = 1$ then we plug this into our general solution

$$1 = x(0) = Ae^{\sin(0)} = A$$

so that $A = 1$. Thus the particular solution is

$$\boxed{x(t) = Ae^{\sin(t)}.}$$

(d) The solution represents a system that oscillates back and forth. However, it is not oscillion as we see with a Hookean spring and a mass system. In this case, this would a oscillation due to something else entirely. Here is a plot of the solution.


4
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12
t

Problem 4. Consider the differential equation

$$x' = \frac{x+t}{t}.$$

(a) Let $f(x,t) = \frac{x+t}{t}$. Show that $f(x,t) = f(\lambda x, \lambda t)$.

(b) Given (a) holds, use the change of variables $u = \frac{x}{t}$ to rewrite the differential equation as a separable equation in terms of u.

(c) Find the general solution to the equation and write your solution in terms of the original variables t and x.

Solution 4.

(a) To show this, we check to see if the equality is true by

$$\begin{aligned} f(\lambda x, \lambda t) &= \frac{\lambda x + \lambda t}{\lambda t} \\ &= \frac{\lambda(x+t)}{\lambda t} \\ &= \frac{x+t}{t} \\ &= f(t). \end{aligned}$$

So the property holds, which leads us to (b).

(b) Now, we let $u = \frac{x}{t}$ which allows us to say $x = tu$. We can then take

$$x' = f(x,t) = f(tu,t) = \frac{tu+t}{t} = \frac{t(u+1)}{t} = u+1.$$

Given our substitution, we can also take

$$x' = (tu)' = u + tu'$$

and substitute this back in our other expression to get

$$u + tu' = u + 1.$$

We can simplify this a bit

$$\begin{aligned} u + tu' &= u + 1 \\ tu' &= 1 \\ u' &= \frac{1}{t}. \end{aligned}$$

This is a separable equation.

(c) Now we can solve the previous equation using separation. So we have

$$\begin{aligned} \frac{du}{dt} &= \frac{1}{t} \\ \int du &= \int \frac{dt}{t} \\ u &= \ln(t) + C. \end{aligned}$$

Recall that we let $u = \frac{x}{t}$ and to get back to the original variable we need to solve for x. So we take

$$\begin{aligned} u = \frac{x}{t} &= \ln(t) + C \\ x &= t\ln(t) + Ct. \end{aligned}$$

So the general solution to our original equation is

$$\boxed{x(t) = t\ln(t) + Ct.}$$

Problem 5. Find the general solution to the following equation.

$$tx' + 2x = \frac{\sin(t)}{t}.$$

Show that your solution is correct. (*Hint: can you use an integrating factor?*)

Solution 5. Note that this is a first order linear equation if we divide the whole expression by t. We can see this by,

$$\begin{aligned} tx' + 2x &= \frac{\sin(t)}{t} \\ x' + \frac{2}{t}x &= \frac{\sin(t)}{t^2}. \end{aligned}$$

This matches the form of a first order linear equation which is typically written as

$$x' + f(t)x = g(t).$$

So note that in our case, $f(t) = \frac{2}{t}$ and $g(t) = \frac{\sin(t)}{t^2}$. Given that, we can solve this equation using the integrating factor technique. For that, we have the integrating factor

$$\begin{aligned} \mu &= e^{\int f(t)dt} \\ &= e^{\int \frac{2}{t}dt} \\ &= e^{2\ln(t)} \\ &= e^{\ln\left(t^2\right)} \\ &= t^2. \end{aligned}$$

Now, we have μ and we can find $x(t)$ by

$$\begin{aligned} x &= \frac{1}{\mu(t)} \int \mu(t)g(t)dt \\ &= \frac{1}{t^2} \int t^2 \frac{\sin(t)}{t^2} dt \\ &= \frac{1}{t^2} \int \sin(t)dt \\ &= -\frac{1}{t^2}\cos(t) + \frac{C}{t^2}. \end{aligned}$$

So our general solution is

$$\boxed{x(t) = -\frac{1}{t^2}\cos(t) + \frac{C}{t^2}.}$$