MATH 271, HOMEWORK 1, *Solutions*

DUE SEPTEMBER 6^{TH}

Problem 1. Look up how to do *integration by parts*. Use this technique to compute the integral

$$\int te^{3t}dt.$$

Solution 1. Integration by parts is a combination of the derivative product rule and the fundamental theorem of calculus. Given functions $u(x)$ and $v(x)$ we can write

$$(uv)' = u'v + uv'.$$

Then if we integrate both sides, we have

$$\int_a^b (uv)'dx = \int_a^b u'vdx + \int_a^b uv'dx.$$

Fundamental theorem of calculus gives us that

$$\int_a^b (uv)'dx = u(b)v(b) - u(a)v(a)$$

and thus

$$\int_a^b u'vdx = u(b)v(b) - u(a)v(a) - \int_a^b uv'dx.$$

Or, without bounds on the integral, we can write

$$\int u'vdx = uv - \int uv'dx.$$

I like to think of integration by parts as shifting the derivative from one function to another with a penalty term. Above, we swap a derivative on u to a derivative on v but have to correct with the function uv. This can all be derived in higher dimensions using Stokes' theorem. It's an excellent tool in the study of differential equations.

Now, the technique to doing integration by parts is to identify a function that when we take its derivative it gets simpler. So, in our integral

$$\int te^{3t}dt$$

we have that t is a function that gets simpler when we take its derivative. That is, $\frac{d}{dt}t = 1$. So, I'll let $u' = e^{3t}$ and $v = t$. Then we have $u = \frac{1}{3}e^{3t}$ and $v' = 1$. Let's replace these into our formula

$$\begin{aligned}
\int u'vdt &= uv - \int uv'dt \\
\int e^{3t}tdt &= \frac{t}{3}e^{3t} - \int \frac{1}{3}e^{3t} \cdot 1dt \\
&= \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} + c \\
&= \left(\frac{t}{3} - \frac{1}{9}\right)e^{3t} + c.
\end{aligned}$$

Problem 2. Convert the following numbers in Cartesian coordinates to polar coordinates and compute all pairwise products.

(a) $z_1 = \frac{1}{2} - \frac{1}{2}i$;

(b) $z_2 = -1 + 3i$;

(c) $z_3 = -2 - 3i$.

Solution 2.

(a) We have that

$$r = \sqrt{z_1 z_1^*} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

and since $a > 0$,

$$\theta = \arctan\left(\frac{1/2}{-1/2}\right) = -\frac{\pi}{4}.$$

So

$$z_1 = \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}.$$

(b) We have $a < 0$ so

$$r = \sqrt{10} \qquad \text{and} \qquad \theta = \arctan\left(\frac{3}{-1}\right) + \pi$$

giving us that

$$z_2 \approx \sqrt{10} e^{1.893i}.$$

(c) Again, $a < 0$ so we have,

$$r = \sqrt{13} \qquad \text{and} \qquad \theta = \arctan\left(\frac{-3}{-2}\right) + \pi.$$

Hence,

$$z_3 \approx \sqrt{13} e^{4.124i}.$$

Now we can compute all pairwise products.

$$\begin{aligned} z_1 z_2 &= 1 + 2i \approx 2.236 e^{1.107i} \\ z_1 z_3 &= \frac{1}{2} - \frac{5}{2}i \approx 2.550 e^{-1.373i} \\ z_2 z_3 &= 11 - 3i \approx 11.402 e^{-0.266i}. \end{aligned}$$

Problem 3. Find the square roots of $-i$ using a geometrical argument.

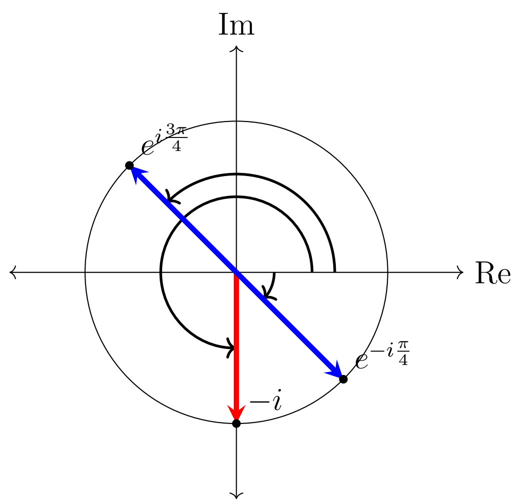
Solution 3. Thinking of $-i = e^{i\frac{3\pi}{2}}$ we can find the square root by finding a number that has a half this rotation in a clockwise way, or a counter clockwise way. So, we have

$$\sqrt{-i} = e^{i\frac{3\pi}{4}}$$

and

$$\sqrt{-i} = e^{-i\frac{\pi}{4}}.$$

Here's a picture to illustrate this.



Problem 4. Draw the unit circle in the complex plane. Plot the complex numbers z_1, z_2, and z_3 given above and find their inverses. Explain what taking the inverse does geometrically.

Solution 4. From the previous problem we had

$$\begin{aligned} z_1 &= \frac{1}{2} - \frac{1}{2}i = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}} \\ z_2 &= -1 + 3i \approx \sqrt{10}e^{-1.893i} \\ z_3 &= -2 - 3i \approx \sqrt{13}e^{4.4124i}. \end{aligned}$$

Recall that the inverse for a complex number $z = a + bi = re^{i\theta}$ is given by

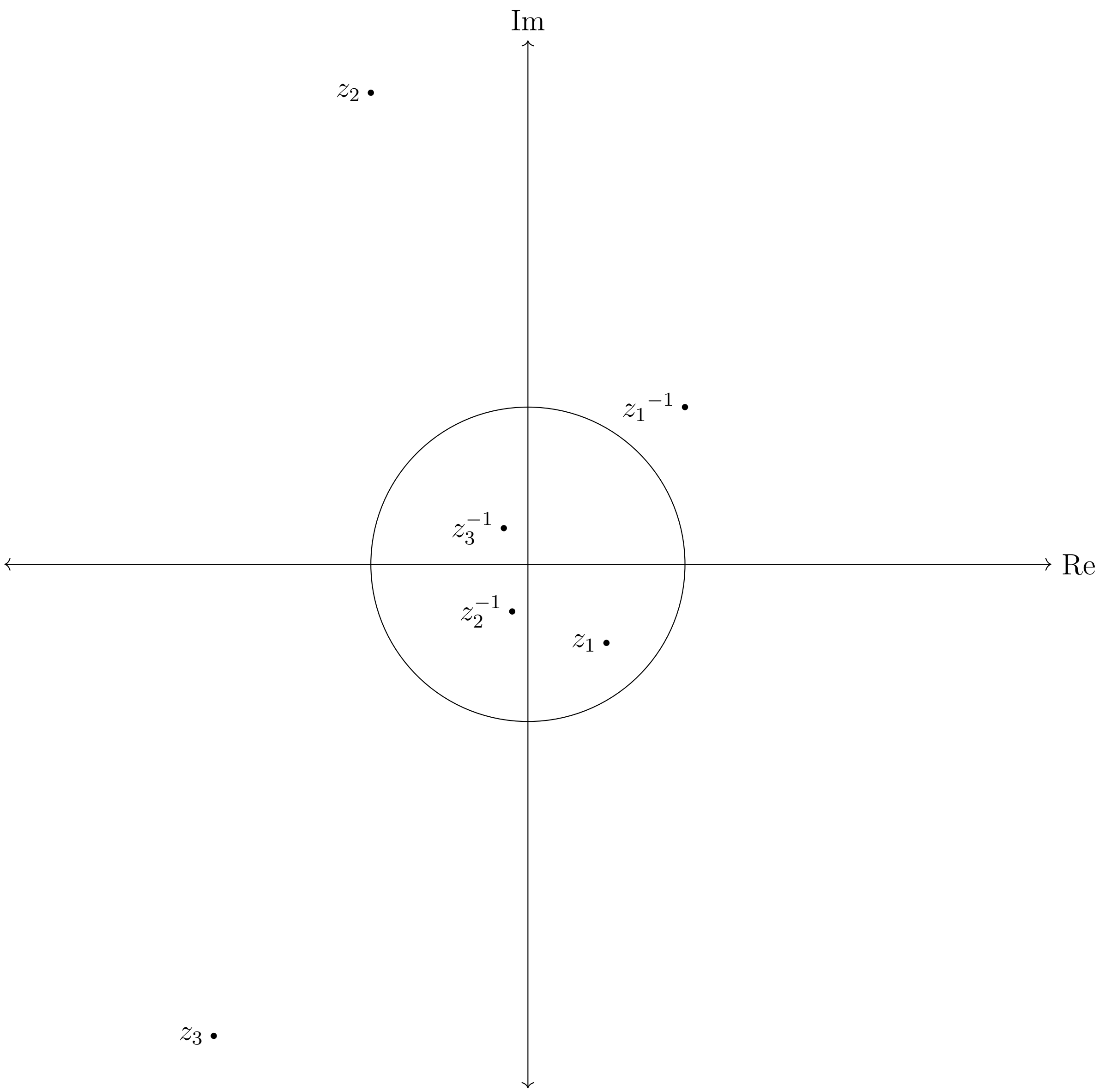
$$z^{-1} = \frac{z^*}{\|z\|^2} = \frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}$$

or in polar coordinates by

$$z^{-1} = \frac{1}{r}e^{-i\theta}.$$

So we have

$$\begin{aligned} z_1^{-1} &= 1 + i = \sqrt{2}e^{i\frac{\pi}{4}} \\ z_2^{-1} &= \frac{-1}{10} - \frac{3i}{10} \approx \frac{1}{\sqrt{10}}e^{1.893i} \\ z_3^{-1} &= \frac{-2}{13} + \frac{3i}{13} \approx \frac{1}{\sqrt{13}}e^{-4.4124i}. \end{aligned}$$



Geometrically what the inverse does is reverses the angle (or argument) θ of the complex number to $-\theta$ and scales the length r by $1/r$.

Problem 5. Look up a differential equation in chemistry that interests you. Write it down, and explain what it attempts to model.

Solution 5. One that interests me is the *Schrödinger equation*. It is useful in describing the motion of subatomic particles and gives rise to the structure of the hydrogen atom. One can then expand a bit to study the helium atom and generalize this further to understand the whole of the periodic table. The equation is, to me, a bit like a square root of a wave equation. The *wave equation* is

$$\nabla^2 f(\mathbf{r}, t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(\mathbf{r}, t),$$

which describes, for example, the vibration of a guitar string or ripples on the surface of a pool. The Schrödinger equation is

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t)\right)\Psi(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}, t).$$

This equation is essentially the quantum version of Newton's law $F = ma$. The equation is very successful at modelling interactions of very small particles such as protons and electrons. The function V is an external potential which could come from, say, an electric field. When $V = 0$, then we have something very similar to a wave equation. The function we wish to solve for is Ψ. However, Ψ itself is not *really* physical. But, it gives rise to statistics which we can measure. For example, if we have a solution Ψ to the equation that describes the position of a particle, then

$$\int_\Omega \Psi^*(\mathbf{r}', t)\Psi(\mathbf{r}', t)d\mathbf{r}'$$

gives the probability of a particle being in the region Ω at time t.

Problem 6. What is a differential equation? What does it mean for a function to be a solution to a differential equation?

Solution 6. A differential equation is an expression containing a function and its derivatives. The idea is that a differential equation relates the rate of change of a function at a point with its values at that point. For example, a function may have a rate of change proportional to its current value which gives rise to

$$x'(t) = kx(t).$$

A function is a solution to a differential equation if, when we plug it in, the expression holds true. The solution may describe the path a particle takes over time, or a how a system chages in space, or many other things really.