Problem 1. Consider the system of linear equations

$$2x - y - 2z = 4$$
$$-x + 3y + 4z = -2$$
$$2x + y - 2z = 8.$$

(a) Write this system of equations as a matrix/vector equation

$$[A]\vec{v} = \vec{b}.$$

(b) Solve the system of equations by finding the inverse of the matrix $[A]$.

Problem 2. Consider the matrix

$$[A] = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors for this matrix.

(b) Construct the matrix $[P]$ such that

$$[\Lambda] = [P]^{-1}[A][P]$$

from the eigenvectors you found.

(c) Find $[P]^{-1}$ and compute

$$[P]^{-1}[A][P].$$

Is this $[\Lambda]$ diagonal?

Problem 3. Consider the matrix

$$[A] = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

(a) Compute the eigenvalues and eigenvectors for the matrix $[A]$.

(b) Show that $\det([A]) = \lambda_1\lambda_2\lambda_3$ and $\text{tr}([A]) = \lambda_1 + \lambda_2 + \lambda_3$ where λ_1, λ_2, and λ_3 are the eigenvalues you found in (a).

(c) Argue why the eigenvalues of $[A]^{-1}$ must be $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$, and $\frac{1}{\lambda_3}$.

Problem 4. For each description below, construct a single 3×3 transformation matrix.

(a) A rotation by $\frac{\pi}{2}$ in the yz-plane.

(b) A rotation by $\frac{\pi}{2}$ in the xy-plane, an interchange of the x and y coordinates.

(c) A matrix which undoes everything in part (b).

Problem 5. A matrix in the group of rotation matrices in $\mathbb{R}^2$ (i.e., SO(2)) can be written as

$$[\text{Rot}]_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for any choice of θ. Another way to think about SO(2) is to consider it as the rotational symmetry group of the unit circle in the plane.

Next, consider a cyclohexane C_6H_{12} molecule:
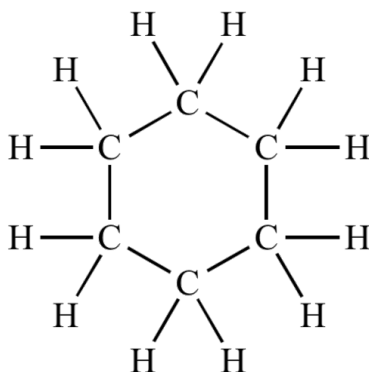
This molecule also has rotational symmetry, but it is a smaller symmetry group than SO(2). We want to determine this rotational symmetry *subgroup*.

(a) This molecule looks much like a hexagon. Determine the external angles of a hexagon.

(b) Note that if we rotate a hexagon (or cyclohexane) by an external angle, then this leaves the molecule invariant (i.e., it looks no different). Using the internal angle you found, write the rotation matrix for that angle and name this matrix $[g]$.

(c) We can *generate* this group C_6 from $[g]$ by repeatedly multiplying $[g]$ with itself. Show that there are only six elements in this group C_6.

(d) These are not all the symmetries of cyclohexane! Explain another symmetry operation that we could use that isn't captured by the rotations above.

If you're interested, look up the group D_{12} which is the *dihedral group of order 12*. Or, taking it further, look at https://en.wikipedia.org/wiki/Cyclohexane_conformation