MATH 255, HOMEWORK 7

Problem 1. Let's examine the idea of level curves and surfaces a bit more. For all of the functions, will consider levels $c_0 = \frac{1}{2}$, $c_1 = 1$, $c_2 = 2$, and $c_3 = 3$.

(a) Given the function

$$f(x) = \frac{1}{\|x\|} = \frac{1}{\sqrt{x^2}},$$

plot this. Then find the *level points* corresponding to c_0, c_1, c_2, and c_3.

(b) Given the function

$$g(x,y) = \frac{1}{\|(x,y)\|} = \frac{1}{\sqrt{x^2+y^2}},$$

plot this. Then find the *level curves* corresponding to c_0, c_1, c_2, and c_3.

(c) Given the function,

$$h(x,y,z) = \frac{1}{\|(x,y,z)\|} = \frac{1}{\sqrt{x^2+y^2+z^2}},$$

find the *level surfaces* corresponding to c_0, c_1, c_2, and c_3. *Note, I didn't ask you to plot h itself since there is not a nice way to do so.*

What's the point? Most functions we care about deal with $\mathbb{R}^3$. However, we don't have ways to visualize these functions without the use of level surfaces. So, working to understand the analogs of level surfaces is key.

Problem 2. Let

$$f(x,y) = x^2 + y^2 - x^2y^2.$$

(a) Compute the equation for the tangent plane at the point $p = (1,2)$.

(b) Compute the gradient of f and find the stationary point(s).

(c) Classify these point(s) as local maxima, local minima, or saddle points.

Problem 3. Let

$$\mathbf{v}(x,y,z) = (x-y, y+x, z)$$

be a vector field in $\mathbb{R}^3$.

(a) Find the Jacobian of this vector field. *Note this quantity is a matrix!*

(b) Compute the determinant of the jacobian at the point $(0,0,0)$.

(c) Write the component functions of $\mathbf{v}$ as follows:

$$\begin{aligned} v_1(x, y, z) &= x - y, \\ v_2(x, y, z) &= y + x, \\ v_3(x, y, z) &= z. \end{aligned}$$

Compute the *divergence* of $\mathbf{v}$

$$\nabla \cdot \mathbf{v} := \frac{\partial}{\partial x} v_1 + \frac{\partial}{\partial y} v_2 + \frac{\partial}{\partial z} v_3.$$

Note this quantity is a scalar!

(d) Compute the *curl* of $\mathbf{v}$

$$\nabla \times \mathbf{v} = \begin{bmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{bmatrix}$$

Note this quantity is a vector!

***Problem 4.** Here we will use Lagrange multipliers to solve a constrained optimization problem. In our case, we want to find where a function $f(x, y)$ is maximal and minimal when subject to a constraint function $g(x, y)$. This type of problem is very common. In fact, this is how we can show that the shape of a red blood cell is optimal for diffusion of oxygen for a given volume! The technique there is just a little bit more advanced.

For us, let's consider the function we want to optimize

$$f(x, y, z) = xyz$$

with the constraint

$$g(x, y, z) = x + y + z = 1$$

and $x, y, z \geq 0$.

This is asking for what rectanglular prism with edges constructed from a given length of wire has the most volume. There will be a few options that you will have to check for which maximizes f.

Problem 5. Now that we are handling functions of more variables, we need to integrate them. Let's consider the functions

$$T(x, y) = 1 + x + y$$

which describes the temperature of a point in the xy-plane and

$$C_p(x, y) = x^2 + y^2.$$

which tells us the *heat capacity* of a point in the xy-plane. We can find the energy contained in a rectangular region $x_0 \leq x \leq x_1$ and $y_0 \leq y \leq y_1$ by

$$E = \int_{y_0}^{y_1} \int_{x_0}^{x_1} T(x, y) C_p(x, y) dx dy.$$

Find the energy in the square region $0 \leq x \leq 2$ and $0 \leq y \leq 2$ for our given functions.