MATH 255, Homework 4: *Solutions*

Relevant Sections: 8.1, 8.2, 8.3, 8.4, 8.5, 8.6

Problem 1. Evaluate the following expressions and simplify to the form $z = a + bi$.

(a) Let $z_1 = 6 + 7i$ and $z_2 = -3 + 3i$. Find $z_1 + z_2$ and $z_1 - z_2$.

(b) Let $z_1 = 5 + 5i$ and $z_2 = -1 + 2i$. Find $z_1 \cdot z_2$ and z_1/z_2.

(c) Take the complex number $z = i + 1$ and multiply by i until you return to your starting point. This should take four iterations.

Solution 1.

(a) We have

$$z_1 + z_2 = (6 + 7i) + (-3 + 3i) = (6 - 3) + i(7 + 3) = 3 + 10i,$$

and

$$z_1 - z_2 = (6 + 7i) - (-3 + 3i) = (6 - (-3)) + i(7 - 3) = 9 + 4i.$$

(b) We have

$$z_1 \cdot z_2 = (5 + 5i) \cdot (-1 + 2i) = -5 - 5i + 10i - 10 = -15 + 5i,$$

and

$$z_1/z_2 = z_1 \cdot z_2^{-1} = (5 + 5i) \cdot \frac{1}{1^2 + 2^2}(-1 - 2i) = \frac{1}{5}(-5 - 5i - 10i + 10) = 1 - 3i.$$

(c) Take

$$i(i + 1) = -1 + i \tag{1}$$
$$i(-1 + i) = -1 - i \tag{2}$$
$$i(-1 - i) = 1 - i \tag{3}$$
$$i(1 - i) = 1 + i. \tag{4}$$

Problem 2. Plot the following points in the complex plane $\mathbb{C}$. Then for each point $z = a+bi$ rewrite in polar form $z = re^{i\theta}$. For each point given in polar form $z = r^{i\theta}$ rewrite it in cartesian form as $z = a + bi$ by using Euler's formula.

(a) From your work on Problem 1 (c) plot and write in polar form the following:

- $z_1 = i + 1$.
- $z_2 = i(i + 1)$.

- $z_3 = i^2(i+1)$.
- $z_4 = i^3(i+1)$.
- $z_5 = i^4(i+1)$.

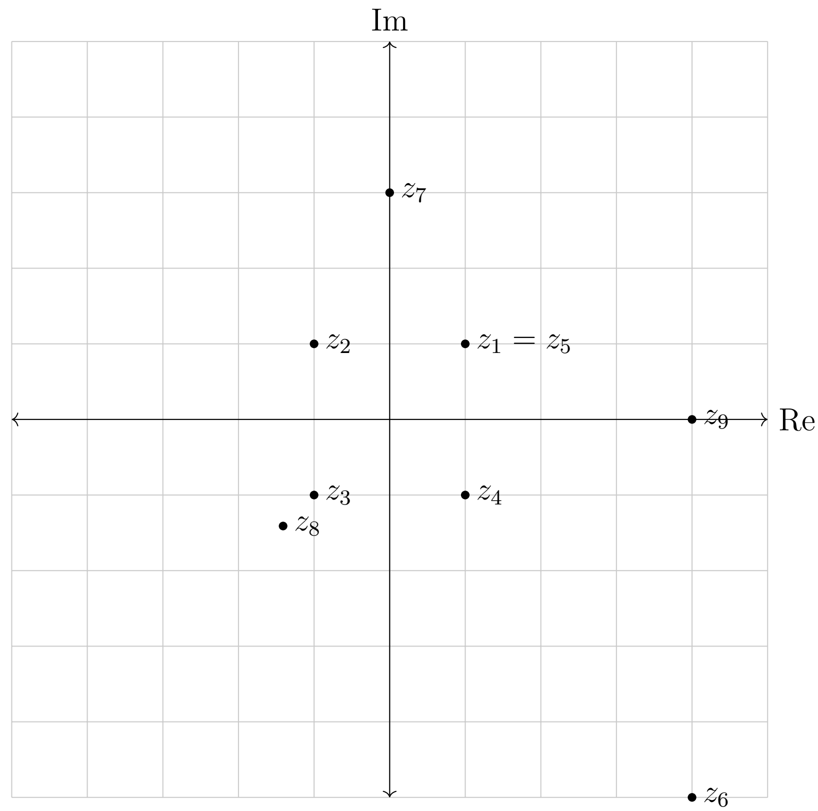
(b) $z_6 = 4 - 5i$.

(c) $z_7 = 3e^{i(\pi/2)}$.

(d) $z_8 = 2e^{i(5\pi/4)}$.

(e) $z_9 = 4e^{i(0)}$.

Solution 2. Here is a plot for each point



(a) We have

- $z_1 = 1 + i = \sqrt{2}e^{i(\pi/4)}$,
- $z_2 = -1 + i = \sqrt{2}e^{i(3\pi/4)}$,
- $z_3 = -1 - i = \sqrt{2}e^{i(5\pi/4)}$,
- $z_4 = 1 - i = \sqrt{2}e^{i(7\pi/4)}$,
- $z_5 = 1 + i = \sqrt{2}e^{i(9\pi/4)} = z_1$.

(b) We have

$$r = \sqrt{4^2 + 5^2} = \sqrt{41},$$

and

$$\theta = \arctan\left(\frac{-5}{4}\right) \approx 0.896.$$

So we can write

$$z_6 = 4 - 5i = \sqrt{41}e^{0.896i}.$$

(c) If we want this number in cartesian form, $a + bi$, then we have

$$a = r\cos\theta = 3\cos\left(\frac{\pi}{2}\right) = 0,$$

and

$$b = r\sin\theta = 3\sin\left(\frac{\pi}{2}\right) = 3.$$

So

$$z_7 = 3e^{i(\pi/2)} = 0 + 3i.$$

(d) Similarly,

$$a = 2\cos\left(\frac{5\pi}{4}\right) = -\sqrt{2},$$

and

$$b = 2\sin\left(\frac{5\pi}{4}\right) = -\sqrt{2}.$$

So

$$z_8 = 2e^{i(5\pi/4)}.$$

(e) Note that $e^{i0} = e^0 = 1$ so

$$z_9 = 4e^{i(0)} = 4.$$

Problem 3. Complex functions (i.e., functions $f\colon \mathbb{C} \to \mathbb{C}$) are tricky to visualize. The issue is that both the input and output are 2-dimensional which means you need some way to visualize 4-dimensional space. For the following, I want you to visit `www.complexgrapher.com` and plot the following functions. Please print these out and attach them to your homework.

(a) $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = z$.

(b) $g\colon \mathbb{C} \to \mathbb{C}$ given by $g(z) = z^2$.

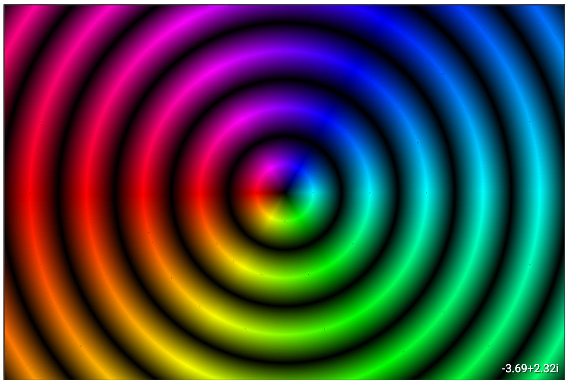
(c) $h\colon \mathbb{C} \to \mathbb{C}$ given by $h(z) = z^3$.

(d) $p\colon \mathbb{C} \to \mathbb{C}$ given by $p(z) = \sin z$.

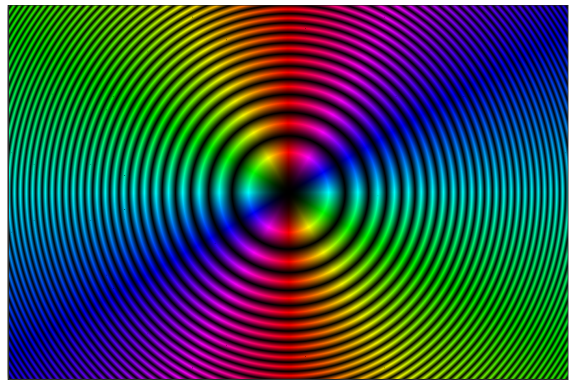
(e) $q\colon \mathbb{C} \to \mathbb{C}$ given by $q(z) = \frac{1}{z^2+1}$.

How does this plotting work? Pick a point $z = a + ib$ on the plane as your input, and if you look at that point, the brightness of each pixel tells you the magnitude r of each complex number and the hue tells you the argument (or angle, or phase) θ of the complex number. Try adjusting the *magnitude modulus*. Adjusting this will give you more of an idea as to what is happening. For example, with the magnitude modulus set to m, you are seeing the remainder of the magnitude r when you divide by m. That is to say, for example, $1 + m$ and 1 will be shown with the same brightness.

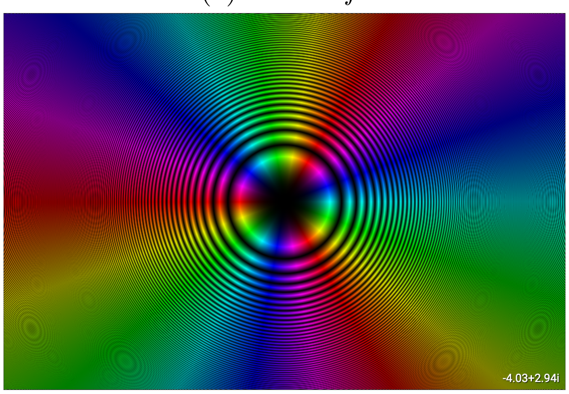
Solution 3.



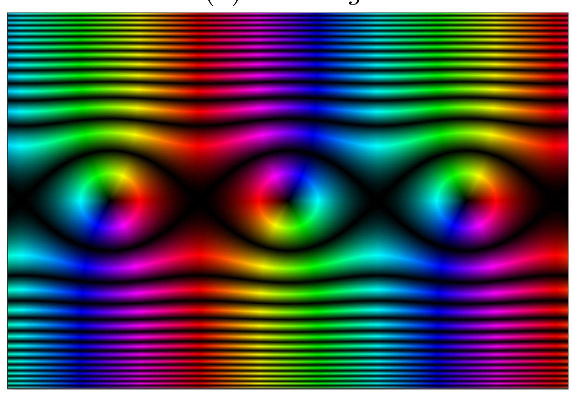
(a) Plot of f.



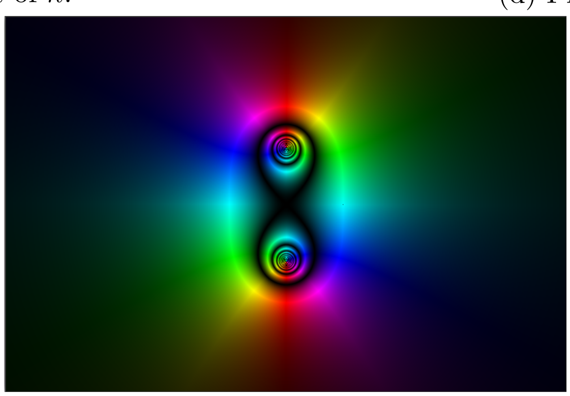
(b) Plot of g.



(c) Plot of h.



(d) Plot of p.



(e) Plot of q.

Problem 4. The point of developing complex numbers is to give us the ability to factor any polynomial. That is, a function of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

By giving us the ability to find $\sqrt{-1}$, we can actually factor any polynomial. Restated, the *fundamental theorem of algebra* says that any polynomial of degree n (the highest power of z in your polynomial) with complex coefficients ($a_i \in \mathbb{C}$) has n complex roots (zeros). For the following, find the roots of the polynomials using WolframAlpha when necessary.

(a) $z^2 + 2$.

(b) $z^3 + z^2 + z + 1$.

(c) $z^4 + z^3 + z^2 + z + 1$.

(d) $z^n - 1$. These are commonly called the *roots of unity*.

Solution 4.

(a) Here we can solve directly,

$$\begin{aligned} z^2 + 2 &= 0 \\ \implies z^2 &= -2 \\ \implies z &= \pm i\sqrt{2}. \end{aligned}$$

(b) Here, we can use WolframAlpha since the cubic formula is awful. We find that the roots are

$$z = -1,\ z = -i,\ z = i.$$

(c) Similarly, the quartic formula is even worse, so we use WolframAlpha. We get

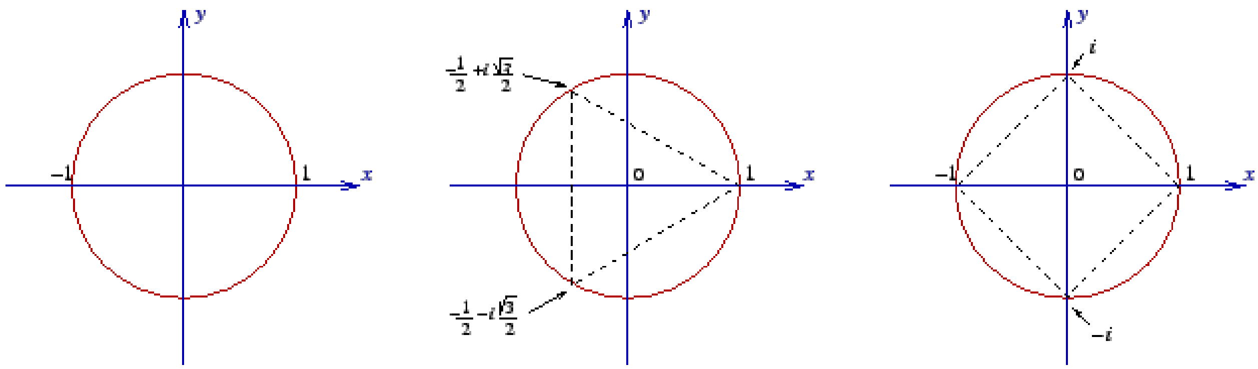
$$z = -(-1)^{1/5},\ z = (-1)^{2/5},\ z = -(-1)^{3/5},\ z = (-1)^{4/5}.$$

But what do these even mean? We don't know what it means to take take these powers here.

(d) Here, we can see a bit more about what these powers really mean by investigating a special case. Namely, we want all the numbers whose nth power is 1 since

$$\begin{aligned} z^n - 1 &= 0 \\ \implies z^n &= 1. \end{aligned}$$

The idea here is that all complex numbers of length 1 are merely just rotations. So we want to break down the rotation of 2π into equal increments. See the picture below for $n = 2$, $n = 3$, and $n = 4$



(c) $n = 2$ (d) $n = 3$ (e) $n = 4$

This is the type of question that you should Google and try to read about in order to find the solution. I'd argue that I didn't teach you quite enough for you to easily go about doing this, but you can spend the time to read some on this and see what is done. See `https://en.wikipedia.org/wiki/Root_of_unity`. Here is the solution, though:

$$z = e^{\frac{i2\pi m}{n}} \text{ for } m = 0, 1, 2, \ldots, n-1.$$

Problem 5. We ran into an issue previously with finding eigenvalues for the following matrix:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Now we have the tools to solve this. Show that the eigenvalues are $\pm i$ and that the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Recall that this matrix was one that rotates vectors in the plane by $\pi/2 = 90°$. The remarkable fact is that the eigenvalues being $\pm i$ capture this same phenomenon. If you look at what happens in Problem 1 and 2 you can see that multiplication of a complex number by i acts like rotation of a vector in the plane.

Solution 5. To find the eigenvalues, we solve

$$\det(A - \lambda I) = 0.$$

So, we get

$$A - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}.$$

Then

$$\det(A - \lambda I) = \lambda^2 + 1 = 0$$

has solutions $\lambda_1 = i$ and $\lambda_2 = -i$.

For λ_1: We find the eigenvectors by

$$(A - iI)\mathbf{v}_1 = \mathbf{0}$$

Which we can make into an augmented matrix

$$M = \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right].$$

Notice $R2$ is i times $R1$ (things are a bit more complicated since we can use complex numbers now), so we have

$$M = \left[\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which letting the $y = 1$, gives us $x = i$. So

$$\mathbf{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

<u>For λ_2:</u> We find the eigenvectors by

$$(A + iI)\mathbf{v}_2 = \mathbf{0}$$

Which we can make into an augmented matrix

$$M = \left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right].$$

Notice $R1$ is i times $R2$ (things are a bit more complicated since we can use complex numbers now), so we have

$$M = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & i & 0 \end{array}\right],$$

which letting the $y = 1$, gives us $x = -i$. So

$$\mathbf{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$