MATH 255, HOMEWORK 3: *Solutions*

Relevant Sections: 17.4, 17.5, 17.6, 18.4, 18.2, 18.6

Problem 1. Consider the system of linear equations:

$$\begin{aligned} 3x + 2y + 0z &= 5 \\ 1x + 1y + 1z &= 3 \\ 0x + 2y + 2z &= 4. \end{aligned}$$

(a) Write the augmented matrix M for this system of equations.

(b) Use row reduction to get the augmented matrix in row-echelon form.

(c) Determine the solution to the system of equations.

Solution 1. (a) We have

$$M = \left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 1 & 1 & 1 & 3 \\ 0 & 2 & 2 & 4 \end{array}\right].$$

(b) We row reduce to the identity matrix on the left side of the bar. So

$$\text{Replace } R_2 \text{ with } R_2 - 1/2R_3 \implies M = \left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 4 \end{array}\right]$$

$$\text{Replace } R_1 \text{ with } R_1 - 3R_2 \implies M = \left[\begin{array}{ccc|c} 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\text{Replace } R_3 \text{ with } R_3 - R_1 \implies M = \left[\begin{array}{ccc|c} 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\text{Swap } R_1 \text{ and } R_2 \implies M = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\text{Divide } R_2 \text{ and } R_3 \text{ by } 2 \implies M = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(c) The above row reduction gives us the system

$$\begin{aligned} 1x + 0y + 0z &= 1 \\ 0x + 1y + 0z &= 1 \\ 0x + 0y + 1z &= 1 \end{aligned}$$

which means that $x = y = z = 1$.

Problem 2. Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 13 \end{bmatrix}.$$

(a) Compute $\det(A)$ and determine whether the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Create an augmented matrix M for this system of equations.

(c) Determine the solution to the system of equations.

Solution 2. (a) To see whether this inhomogenous system has a unique solution, we take

$$\det(A) = 1.$$

Since $\det(A) = 1 \neq 0$ we know that there exists a unique solution. (*There are propositions for inhomogenous and homogeneous equations in the notes. Be sure to take note of these.*)

(b) We have

$$M = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1/2 & 7/2 \end{array}\right].$$

(c) We perform row reduction to get the identity on the left side of the bar in M. I omit the steps here but you find that $x = 2$, $y = -5$, and $z = 7$ so that

$$\mathbf{x} = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}.$$

Problem 3. Find the inverse matrix for each of the following:

(a)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(b)

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution 3. (a) We create the augmented matrix

$$M = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right].$$

Then we row reduce the left hand side of the bar to the identity matrix.

$$\text{Swap } R_1 \text{ and } R_2 \implies \left[\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\text{Multiply } R_1 \text{ by } -1 \implies \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

So we have

$$A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We can verify this by

$$A^{-1}A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) We repeat this process again. First we make

$$M = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

We then row reduce the left side of the bar to the identity matrix. I omit the work here, but in the end you have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right].$$

So we have

$$B^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}.$$

You can perform the same check as we did for the previous part.

Problem 4. Construct transformation matrices that represent the following rotations about the z-axis:

(a) Counterclockwise through $45^\circ = \frac{\pi}{4}$.

(b) Counterclockwise through $90^\circ = \frac{\pi}{2}$.

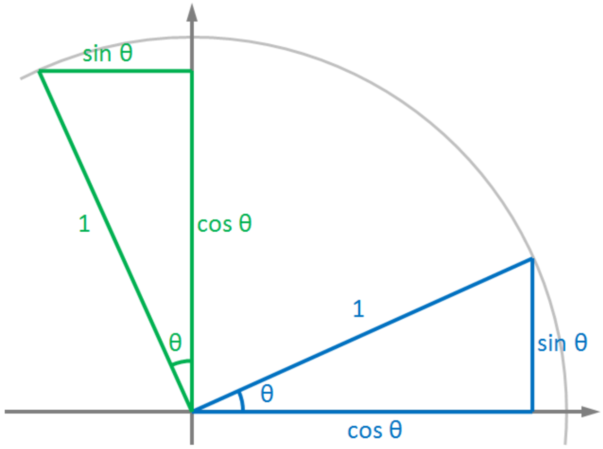
(c) Clockwise through $90^\circ = \frac{\pi}{2}$.

(*Hint: This necessary matrix is given to you in the notes and in the book, chapter 18).*

Solution 4. The book provides us the matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

which rotates a vector in the plane by an angle θ in the counterclockwise direction. However, it is worth looking to see how this is derived (though I did not want you to derive this yourself).



Recall from Homework 2 Problem 4 that we can understand a linear transformation entirely by how it transforms the basis vectors. In this case, we want

$$R\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \qquad R\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

to each be rotated by an angle θ. So we draw the following Here we notice that

$$R\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

and

$$R\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}.$$

This leads us exactly to

$$R = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

(a) We plug in $\theta_a = \pi/4$ and find

$$R_a = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

(b) We plug in $\theta_b = \pi/2$ and find

$$R_b = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

(c) We plug in $\theta_c = -\pi/2$ since a negative angle will rotate clockwise. Then we get

$$R_c = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

Problem 5. Find the eigenvalues and eigenvectors for the following matrices.

(a)

$$A = \begin{bmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix}.$$

(b)

$$B = \begin{bmatrix} -1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ -1 & 1 & 0 \end{bmatrix}.$$

Solution 5. (a) First, we take

$$\det(A - \lambda I) = \begin{vmatrix} 5/2 - \lambda & 1/2 \\ 1/2 & 5/2 - \lambda` \end{vmatrix} = (5/2 - \lambda)^2 - 1/4.$$

We set this equal to zero and simplify to find

$$\begin{aligned} \lambda^2 + 5\lambda + 6 &= 0 \\ (\lambda - 2)(\lambda - 3) &= 0 \end{aligned}$$

meaning that we have $\lambda_1 = 2$ and $\lambda_2 = 3$. Then we find each eigenvector.

For $\lambda_1 = 2$:

We solve

$$(A - 2I)\mathbf{v}_1 = \mathbf{0}.$$

Writing this out as an augmented matrix, we get

$$M = \left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right]$$

Row reducing as much as we can gives us

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and the system of equations

$$\begin{aligned} 1x + 1y &= 0 \\ 0x + 0y &= 0. \end{aligned}$$

The second equation means that y is a free variable. I'll choose $y = 1$. Then we get that $x = -y$ from the first equation. So we have that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Of course, you can choose any nonzero value for y.

For $\lambda_2 = 3$:

We solve

$$(A - 3I)\mathbf{v}_2 = \mathbf{0}.$$

Writing this out as an augmented matrix, we get

$$M = \left[\begin{array}{cc|c} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{array}\right]$$

Row reducing as much as we can gives us

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and the system of equations

$$\begin{aligned} -1x + 1y &= 0 \\ 0x + 0y &= 0. \end{aligned}$$

The second equation means that y is a free variable. I'll choose $y = 1$. Then we get that $x = y$ from the first equation. So we have that

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Fun fact: If we take the eigenvectors and place them in a matrix

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then we can write

$$A = PDP^{-1}$$

where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

is a diagonal matrix containing the eigenvalues of A along the diagonal. This is where the name *diagonalization* comes from. This process can be generalized and it yields what is called the *singular value decomposition* or SVD. SVD is widely used in data analysis and is an extremely important result of linear algebra!

(b) We repeat the usual process here. First

$$\begin{aligned} \det(B - \lambda I) &= 0 \\ \implies \lambda - \lambda^3 &= 0 \\ \implies \lambda(\lambda - 1)(\lambda + 1) &= 0. \end{aligned}$$

So the eigenvalues are

$$\lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = -1.$$

I omit the work here, but the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$