MATH 255, Homework 2: *Solutions*

Relevant Sections: 18.1, 18.3, 17.2, 17.2, 17.3.

Problem 1. Which of the following are linear transformations? For those that are not, which properties of *linearity* (the properties (i), (ii), and (iii) in our notes) fail? Show your work.

(a) $T_a \colon \mathbb{R} \to \mathbb{R}$ given by $T_a(x) = \frac{1}{x}$.

(b) $T_b \colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T_b \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

(c) $T_c \colon \mathbb{R} \to \mathbb{R}^3$ given by

$$T_c(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}.$$

(d) $T_d \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T_d \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x+y \\ x+y \\ x+y \end{bmatrix}.$$

Solution 1. The three checks we make to see if T is linear are

(i) $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$;

(ii) $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$;

(iii) $T(\mathbf{0}) = \mathbf{0}$.

Logically, (i) or (ii) imply (iii). However, (iii) is a nice quick check for linearity.

(a) This function is nonlinear. To see this, let us compare the Left Hand Side (LHS) with the Right Hand Side (RHS).

(i) LHS:

$$T_a(x+y) = \frac{1}{x+y}.$$

RHS:

$$T_a(x) + T_a(y) = \frac{1}{x} + \frac{1}{y}.$$

Clearly we have LHS$\neq$RHS. Just take $x = y = 1$.

(ii) <u>LHS:</u>

$$T_a(\lambda x) = \frac{1}{\lambda x}.$$

<u>RHS:</u>

$$\lambda T_a(x) = \frac{\lambda}{x}.$$

So LHS≠RHS.

(iii) We cannot even consider 1/0 as this is not well-defined. Clearly (iii) does not hold.

(b) This function is linear.

(i) <u>LHS:</u>

$$T_b(\mathbf{v}+\mathbf{w}) = T_b\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T_b\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}.$$

<u>RHS:</u>

$$T_b(\mathbf{v}) + T_b(\mathbf{w}) = T_b\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T_b\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}.$$

So the LHS=RHS.

(ii) <u>LHS:</u>

$$T_b(\lambda\mathbf{v}) = T_b\left(\lambda\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T_b\left(\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}\right) = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}.$$

<u>RHS:</u>

$$\lambda T_b(\mathbf{v}) = \lambda T_b\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \lambda\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}.$$

So LHS=RHS.

(iii) We have

$$T_b(\mathbf{0}) = T_b\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Notice that these are the $\mathbf{0}$ in different dimensional vector spaces (i.e., $\mathbb{R}^3$ and $\mathbb{R}^2$). This is allowed. Just understand that changing the dimension does not change the idea of what we consider to be the origin. Maybe we should denote the input $\mathbf{0}_3$ and the output $\mathbf{0}_2$. However, it is really unimportant to us at this moment.

(c) This function is nonlinear.

(i) LHS:

$$T_c(t_1+t_2) = \begin{bmatrix} t_1+t_2 \\ (t_1+t_2)^2 \\ (t_1+t_2)^3 \end{bmatrix}.$$

RHS:

$$T_c(t_1)+T_c(t_2) = \begin{bmatrix} t_1 \\ t_1^2 \\ t_1^3 \end{bmatrix} + \begin{bmatrix} t_2 \\ t_2^2 \\ t_2^3 \end{bmatrix} = \begin{bmatrix} t_1+t_2 \\ t_1^2+t_2^2 \\ t_1^3+t_2^3 \end{bmatrix}.$$

So LHS≠RHS. Just take $t_1 = t_2 = 1$ to see this.

(ii) LHS:

$$T_c(\lambda t) = \begin{bmatrix} \lambda t \\ (\lambda t)^2 \\ (\lambda t)^3 \end{bmatrix}.$$

RHS:

$$\lambda T_c = \lambda \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix} = \begin{bmatrix} \lambda t \\ \lambda t^2 \\ \lambda t^3 \end{bmatrix}.$$

So LHS≠RHS.

(iii) Take

$$T_c(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In this case (iii) holds while (i) and (ii) do not.

(d) This function is linear.

(i) LHS:

$$T_d(\mathbf{v}+\mathbf{w}) = T_d\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T_d\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+y_1+y_2 \\ x_1+x_2+y_1+y_2 \\ x_1+x_2+y_1+y_2 \end{bmatrix}.$$

RHS:

$$T_d(\mathbf{v})+T_d(\mathbf{w}) = T_d\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) T_d\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+y_1 \\ x_1+y_1 \\ x_1+y_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ x_2+y_2 \\ x_2+y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2+y_1+y_2 \\ x_1+x_2+y_1+y_2 \\ x_1+x_2+y_1+y_2 \end{bmatrix}.$$

So the LHS=RHS.

(ii) LHS:

$$T_d(\lambda\mathbf{v}) = T_d\left(\lambda \begin{bmatrix} x \\ y \end{bmatrix}\right) = T_d\left(\begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}\right) = \begin{bmatrix} \lambda x+\lambda y \\ \lambda x+\lambda y \\ \lambda x+\lambda y \end{bmatrix}.$$

RHS:

$$\lambda T_d(\mathbf{v} = \lambda T_d\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \lambda \begin{bmatrix} x+y \\ x+y \\ x+y \end{bmatrix} = \begin{bmatrix} \lambda x + \lambda y \\ \lambda x + \lambda y \\ \lambda x + \lambda y \end{bmatrix}.$$

So the LHS=RHS.

(iii) Take

$$T_d(\mathbf{0}) = T_d\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So (iii) also holds.

Problem 2. Write down the matrix for the following linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ 2x \\ 3y+z \end{bmatrix}.$$

Solution 2. A linear transformation and left multiplication of a vector by a matrix are analogous. What I'm saying here is to find a matrix

$$T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

so that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y+z \\ 2x \\ 3y+z \end{bmatrix}.$$

We do the matrix multiplication on the left hand side and I'll rewrite the right hand side slightly to get

$$\begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix} = \begin{bmatrix} 1x+1y+1z \\ 2x+0y+0z \\ 0x+3y+1z \end{bmatrix}.$$

Notice this gives us the system of equations that allow us to solve for the a_{ij}. Namely,

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= 1x+1y+1z \\ a_{21}x + a_{22}y + a_{23}z &= 2x+0y+0z \\ a_{31}x + a_{32}y + a_{33}z &= 0x+3y+1z. \end{aligned}$$

The coefficients infront of the x, y, and z must match on each line which leads us to

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

You can double check this by performing the matrix multiplication again.

Problem 3. Compute the following:

(a)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

(b)

$$\mathbf{B} = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(c)

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 3 \end{bmatrix}$$

(d) Take

$$\mathbf{M} = \begin{bmatrix} 10 & 15 \\ 20 & 10 \end{bmatrix}$$

and

$$\mathbf{N} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Find $3\mathbf{MN} - 3\mathbf{NM}$.

Solution 3. We just multiply these out.

(a) We have a 1×3 on a 3×1. So we expect a 1×1 output.

$$\mathbf{A} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}.$$

(b) We have a 2×3 on a 3×1. So we expect a 2×1 output.

$$\mathbf{B} = \begin{bmatrix} 5 \cdot 3 + 0 \cdot 2 + 0 \cdot 1 \\ 2 \cdot 3 + 2 \cdot 2 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}.$$

(c) We have a 3×4 on a 4×2. So we expect a 3×2 output.

$$\mathbf{C} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 2 & 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 3 \\ 5 \cdot 2 + 6 \cdot 2 + 7 \cdot 3 + 8 \cdot 2 & 5 \cdot 2 + 6 \cdot 3 + 7 \cdot 2 + 8 \cdot 3 \\ 9 \cdot 3 + 10 \cdot 2 + 11 \cdot 3 + 12 \cdot 2 & 9 \cdot 2 + 10 \cdot 3 + 11 \cdot 2 + 12 \cdot 3 \end{bmatrix} = \begin{bmatrix} 24 & 26 \\ 64 & 66 \\ 104 & 106 \end{bmatrix}.$$

(d) First, note that in general for two matrices A and B that $AB \neq BA$. So we cannot a priori assume $3\mathbf{MN} - 3\mathbf{NM} = 0$. Second, we can rewrite

$$3\mathbf{MN} - 3\mathbf{NM} = 3(\mathbf{MN} - \mathbf{NM}).$$

(*Aside: The quantity* $\mathbf{MN} - \mathbf{NM}$ *is sometimes written* $[\mathbf{M}, \mathbf{N}]$ *and is called the commutator. This relationship is necessary to understand in quantum mechanics!*)

We compute

$$\mathbf{MN} = \begin{bmatrix} 10 & 15 \\ 20 & 10 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 \cdot 1 + 15 \cdot 2 & 10 \cdot 2 + 15 \cdot 1 \\ 20 \cdot 1 + 10 \cdot 2 & 20 \cdot 2 + 10 \cdot 1 \end{bmatrix} = \begin{bmatrix} 40 & 35 \\ 40 & 50 \end{bmatrix}$$

and

$$\mathbf{NM} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 10 & 15 \\ 20 & 10 \end{bmatrix} = \begin{bmatrix} 1 \cdot 10 + 2 \cdot 20 & 1 \cdot 15 + 2 \cdot 10 \\ 2 \cdot 10 + 1 \cdot 20 & 2 \cdot 15 + 1 \cdot 10 \end{bmatrix} = \begin{bmatrix} 50 & 35 \\ 40 & 40 \end{bmatrix}.$$

Then we have

$$3(\mathbf{MN} - \mathbf{NM}) = 3 \cdot \left(\begin{bmatrix} 40 & 35 \\ 40 & 50 \end{bmatrix} - \begin{bmatrix} 50 & 35 \\ 40 & 40 \end{bmatrix} \right) = 3 \cdot \begin{bmatrix} -10 & 0 \\ 0 & 30 \end{bmatrix} = \begin{bmatrix} -30 & 0 \\ 0 & 30 \end{bmatrix}.$$

Problem 4. Compute the following determinants:

(a)

$$\det(\mathbf{A}) = \begin{vmatrix} -3 & 6 \\ -3 & 6 \end{vmatrix}$$

(b)

$$\det(\mathbf{B}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(c)

$$\det(\mathbf{C}) = \begin{vmatrix} \lambda & 2 & 0 \\ 0 & \lambda - 1 & 5 \\ 0 & 0 & \lambda \end{vmatrix}$$

Solution 4. We just compute.

(a)

$$\det(\mathbf{A}) = (-3 \cdot 6) - (6 \cdot (-3)) = 0.$$

(b) Expanding across the top row, we have

$$\begin{aligned} \det(\mathbf{B}) &= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) \\ &= -3 - 2(-6) + 3(-3) = 0. \end{aligned}$$

(c) Expanding across the left column, we have

$$\begin{aligned} \det(\mathbf{C}) &= \lambda \cdot \begin{vmatrix} \lambda - 1 & 5 \\ 0 & \lambda \end{vmatrix} - 0 \begin{vmatrix} 2 & 0 \\ 0 & \lambda \end{vmatrix} + 0 \begin{vmatrix} 2 & 0 \\ \lambda - 1 & 5 \end{vmatrix} \\ &= \lambda((\lambda - 1) \cdot \lambda - 5 \cdot 0) = \lambda^2(\lambda - 1). \end{aligned}$$

Problem 5. A linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

(a) Compute how T transforms the standard basis elements for $\mathbb{R}^3$. That is, find

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

(b) If we apply this linear transformation to the unit cube (that is, all points who have (x, y, z) coordinates with $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$), what will the volume of the transformed cube be? (*Hint: the determinant of this matrix* **T** *provides us this information.*)

Solution 5. (a) The moral of the story here is that we can understand a linear transformation entirely by how it transforms the standard basis vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Recall that a linear transformation T is given by left multiplying by the matrix $\mathbf{T}$ above. So

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 0 + 0 \cdot 0 \\ 2 \cdot 1 + 1 \cdot 0 + 2 \cdot 0 \\ 0 \cdot 1 + 2 \cdot 0 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 2 \cdot 1 + 0 \cdot 0 \\ 2 \cdot 0 + 1 \cdot 1 + 2 \cdot 0 \\ 0 \cdot 0 + 2 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 2 \cdot 0 + 0 \cdot 1 \\ 2 \cdot 0 + 1 \cdot 0 + 2 \cdot 1 \\ 0 \cdot 0 + 2 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Notice that, for example,

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

is the first column of the matrix $\mathbf{T}$. We see an analogous result for the second and third basis vectors. This may help you understand matrix multiplication and linear transformations just a bit more.

(b) We have that

$$\det(\mathbf{T}) = -7.$$

I did not show work here. The negative is relatively unimportant for this example since we're just trying to understand how volume is transformed by T. So it turns out in this case, a cube of $1[m^3]$ would be transformed into a parallelopiped with a volume of $7[m^3]$. The minus sign has to do with an "orientation." Let us not worry about this right now. The idea of how volume is transformed when we apply a transformation will be of utmost importance when we begin integration in 3-dimensional space.