MATH 255, Homework 2

Relevant Sections: 18.1, 18.3, 17.2, 17.2, 17.3.

Problem 1. Which of the following are linear transformations? For those that are not, which properties of *linearity* (the properties (i), (ii), and (iii) in our notes) fail? Show your work.

(a) $T_a\colon \mathbb{R} \to \mathbb{R}$ given by $T_a(x) = \frac{1}{x}$.

(b) $T_b\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T_b\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

(c) $T_c\colon \mathbb{R} \to \mathbb{R}^3$ given by

$$T_c(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}.$$

(d) $T_d\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T_d\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x+y \\ x+y \end{bmatrix}.$$

Problem 2. Write down the matrix for the following linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ 2x \\ 3y+z \end{bmatrix}.$$

Problem 3. Compute the following:

(a)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

(b)

$$\mathbf{B} = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(c)

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 3 \end{bmatrix}$$

(d) Take

$$\mathbf{M} = \begin{bmatrix} 10 & 15 \\ 20 & 10 \end{bmatrix}$$

and

$$\mathbf{N} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Find $3\mathbf{MN} - 3\mathbf{NM}$.

Problem 4. Compute the following determinants:

(a)

$$\det(\mathbf{A}) = \begin{vmatrix} -3 & 6 \\ -3 & 6 \end{vmatrix}$$

(b)

$$\det(\mathbf{B}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(c)

$$\det(\mathbf{C}) = \begin{vmatrix} \lambda & 2 & 0 \\ 0 & \lambda - 1 & 5 \\ 0 & 0 & \lambda \end{vmatrix}$$

Problem 5. A linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

(a) Compute how T transforms the standard basis elements for $\mathbb{R}^3$. That is, find

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right).$$

(b) If we apply this linear transformation to the unit cube (that is, all points who have (x, y, z) coordinates with $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$), what will the volume of the transformed cube be? (*Hint: the determinant of this matrix* $\mathbf{T}$ *provides us this information.*)