MATH 255, Homework 11: *Solutions*

Problem 1 and 2 are related.

Problem 1. Consider the following linear system

$$x'(t) = x - y$$
$$y'(t) = -x - y.$$

(a) Rewrite this as a matrix equation

$$\mathbf{v}' = M\mathbf{v}.$$

Here the vector $\mathbf{v}$ denotes the xy-position of a particle at time t.

(b) Plot the vector field $\mathbf{v}'$.

(c) Describe what happens if your initial data is

i. $(x_0, y_0) = (0, 0)$,

ii. $(x_0, y_0) = (1, 1)$,

iii. $(x_0, y_0) = (-1, -1)$.

Solution 1.

(a) We let

$$\mathbf{v} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{and} \quad \mathbf{v}' = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}.$$

So we want an equation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we multiply the matrices on the right hand side, we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} M_{11}x + M_{12}y \\ M_{21}x + M_{22}y \end{bmatrix}.$$

This gives us the system of equations

$$x' = M_{11}x + M_{12}y$$
$$y' = M_{21}x + M_{22}y.$$

We match these matrix coefficients with our given system and find

$$M_{11} = 1 \qquad M_{12} = -1$$
$$M_{21} = -1 \qquad M_{22} = -1,$$

and we put

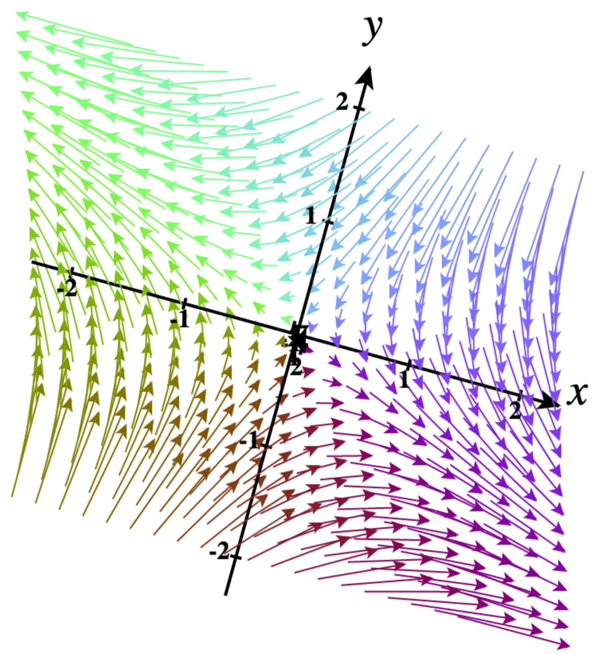
$$M = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Finally, our matrix equation reads

$$\begin{bmatrix} x' \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(b) Here is a plot of the vector field

$$\mathbf{v}' = \begin{bmatrix} x - y \\ -x - y \end{bmatrix}.$$



(c)

i. If the starting point is $(x_0, y_0) = (0, 0)$ then we have

$$x' = 0 - 0 = 0$$
$$y' = -0 - 0 = 0,$$

so both $x' = y' = 0$. Thus, if we start at the origin, we stay at the origin.

ii. If we start at $(x_0, y_0) = (1, 1)$ then we have

$$x' = 1 - 1 = 0$$
$$y' = -1 - 1 = -2$$

so we have no initial movement in the x-direction and only negative movement in the y-direction. However, it seems over time that the x-values grow and the y-values continue to decrease.

iii. This point is similar the previous as we have $x' = 0$ initially but y' is positive instead of negative. In this case we an see that we are carried in the negative x-direction and the positive y-direction over time.

***Problem 2.** With the same linear system as in 1, do the following.

(a) Compute the eigenvalues of the matrix M.

(b) Compute the eigenvectors of the matrix M.

(c) Write the general solution for this system.

(d) Find the particular solution corresponding to the initial data $(x(0), y(0)) = (1, 1)$.

Solution 2.

(a) The eigenvalues are found by solving

$$\det(M - \lambda I) = 0.$$

So we have

$$\begin{aligned} \det(M - \lambda I) &= 0 \\ \left| \begin{bmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{bmatrix} \right| &= 0 \\ (1-\lambda)(-1-\lambda) - 1 &= 0 \\ -1 + \lambda - \lambda + \lambda^2 - 1 &= 0 \\ \lambda^2 - 2 &= 0 \\ \lambda^2 &= 2 \\ \implies \lambda &= \pm\sqrt{2}. \end{aligned}$$

We'll set $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2}$.

(b) Now, to find the eigenvectors, we want to find a vector $\mathbf{e}$ that solves

$$(M - \lambda)\mathbf{e} = \mathbf{0}.$$

<u>For $\lambda_1 = \sqrt{2}$</u> We have,

$$\begin{bmatrix} 1-\sqrt{2} & -1 \\ -1 & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can make the augmented matrix, and solve by row reduction. So we have

$$\left[\begin{array}{cc|c} 1-\sqrt{2} & -1 & 0 \\ -1 & -1-\sqrt{2} & 0 \end{array}\right] \xrightarrow{\text{Add R2 from R1}} \left[\begin{array}{cc|c} -\sqrt{2} & -2-\sqrt{2} & 0 \\ -1 & -1-\sqrt{2} & 0 \end{array}\right]$$

$$\xrightarrow{\text{Multiply R2 by } \sqrt{2}} \left[\begin{array}{cc|c} -\sqrt{2} & -2-\sqrt{2} & 0 \\ -\sqrt{2} & -2-\sqrt{2} & 0 \end{array}\right] \xrightarrow{\text{Subtract R1 from R2}} \left[\begin{array}{cc|c} -\sqrt{2} & -2-\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

This gives us the equation

$$\begin{aligned} -\sqrt{2}x + (-2-\sqrt{2})y &= 0 \\ -\sqrt{2}x &= (2+\sqrt{2})y \\ x &= \left(\frac{-2}{\sqrt{2}} - 1\right) y \\ x &= (-1-\sqrt{2})y. \end{aligned}$$

So choose $y = 1$ and we find that the eigenvector corresponding to λ_1 is

$$\mathbf{e}_1 = \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix}.$$

For $\lambda_2 = -\sqrt{2}$ The work is very similar, and you find that the corresponding eigenvector is

$$\mathbf{e}_2 = \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}.$$

(c) The general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{\sqrt{2}t} \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix} + c_2 e^{-\sqrt{2}t} \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}.$$

(d) We use the initial data so we know

$$\begin{aligned} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= c_1 e^{\sqrt{2}\cdot 0} \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix} + c_2 e^{-\sqrt{2}\cdot 0} \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} c_1(-1-\sqrt{2}) \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2(-1+\sqrt{2}) \\ c_2 \end{bmatrix}. \end{aligned}$$

This gives us the following system of equations

$$1 = c_1(-1-\sqrt{2}) + c_2(-1+\sqrt{2}) \quad (1)$$

$$1 = c_1 + c_2. \quad (2)$$

Note that (2) gives us that $c_2 = 1 - c_1$ and we plug this into (1) to find

$$\begin{aligned} 1 &= c_1(-1-\sqrt{2}) + (1-c_1)(-1+\sqrt{2}) \\ 1 &= -c_1 - \sqrt{2}c_1 - 1 + \sqrt{2} + c_1 - \sqrt{2}c_1 \\ 2-\sqrt{2} &= -2\sqrt{2}c_1 \\ \implies c_1 &= \frac{1}{2} - \frac{1}{\sqrt{2}}. \end{aligned}$$

Thus, $c_2 = \frac{1}{2} + \frac{1}{\sqrt{2}}$. So the particular solution is

$$\begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix} = \left(\frac{1}{2} - \frac{1}{\sqrt{2}}\right) e^{\sqrt{2}t} \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix} + \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) e^{-\sqrt{2}t} \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$$

Problem 3. Solving the one dimensional Laplace equation is much like an ODE. However, the data given looks a bit different. So consider the following set up.

Consider the Laplace equation

$$\Delta u(x) = \frac{d^2u}{dx^2} = 0$$

on the interval $\Omega = (0, 1)$ with boundary conditions $u(0) = 0$ and $u(1) = 1$.

(a) This equation is separable. To find u, take two antiderivatives of

$$\frac{d^2u}{dx^2} = 0.$$

(b) To verify you did this correctly, take two derivatives of your function to see that you get 0.

(c) Your function should have two undetermined constants. Solve for these constants using the boundary conditions provided.

Solution 3.

(a) We take an antiderivative

$$\begin{aligned} \int \frac{d^2u}{dx^2} dx &= \int 0 dx \\ \frac{du}{dx} &= c_1, \end{aligned}$$

by the fundamental theorem of calculus. We can do this again and get

$$\begin{aligned} \int \frac{du}{dx} dx &= \int c_1 dx \\ u(x) &= c_1 x + c_2. \end{aligned}$$

(b) We check this by taking two derivatives

$$\frac{d}{dx}\frac{d}{dx}(c_1 + x + c_2) = \frac{d}{dx} c_1 = 0.$$

So this function does work.

(c) We know that

$$u(0) = 0 = c_1(0) + c_2$$

which means $c_2 = 0$. Then we also have

$$u(1) = 1 = c_1(1)$$

which means that $c_1 = 1$. So we have

$$u(x) = x.$$

Problem 4. The methods for solving many PDEs are beyond the scope of this class, but we can still see what solutions behave like and a bit of how to find these. What we'll do below are a few steps of the method of *separation of variables* (not to be confused with separable ODE!)

Consider the heat equation in one dimension on the region $\Omega = (0, 1)$

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0$$

with boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$, and initial condition $u(x,0) = \sin(\pi x)$.

(a) Show that $f(x) = \sin(\pi x)$ is a solution to

$$f''(x) = -\pi^2 f(x)$$

with $f(0) = 0$ and $f(1) = 0$.

(b) Show that $g(t) = e^{-\pi^2 t}$ is a solution to

$$g'(t) = -\pi^2 g(t)$$

with $g(0) = 1$.

(c) Show that $u(x,t) = f(x)g(t)$ solves the heat equation with these boundary and initial conditions.

Solution 4.

(a) We have

$$\begin{aligned}\frac{d^2 f}{dx^2} &= \frac{d}{dx}\frac{d}{dx}\sin(\pi x) \\ &= \pi\frac{d}{dx}\cos(\pi x) \\ &= -\pi^2\sin(\pi x) \\ &= -\pi^2 f(x).\end{aligned}$$

So we have the ODE is solved. Then we also check

$$f(0) = \sin(\pi 0) = 0$$

and

$$f(1) = \sin(\pi) = 0$$

and so the boundary conditions are satisfied.

(b) We have

$$\frac{dg}{dt} = \frac{d}{dt}e^{-\pi^2 t} = -\pi^2 e^{-\pi^2 t} = -\pi^2 g(t).$$

Then note that

$$g(0) = e^{-\pi^2 \cdot 0} = 1.$$

So this solves the PDE and initial value.

(c) We have

$$\begin{aligned}\frac{\partial}{\partial t}f(x)g(t) - \frac{\partial^2}{\partial x^2}f(x)g(t) &= f(x)g'(t) - f''(x)g(t)\\ &= -\pi^2 f(x)g(t) + \pi^2 f(x)g(t)\\ &= 0.\end{aligned}$$

So indeed this function $u(x,t)$ does satisfy the PDE. Now we also check

$$u(0,t) = e^{-\pi^2 \cdot 0}\sin(0) = 0$$

and

$$u(1,1) = e^{-\pi^2}\sin(\pi) = 0,$$

so the boundary conditions are satisfied. Lastly, we have

$$u(x,0) = e^{-\pi^2 \cdot 0}\sin(\pi x) = \sin(\pi x),$$

so the initial conditions are satisfied.

Problem 5. With our solution from 4, we can analyze the behavior of the system. The physical phenomenon that Problem 4 modelled was a thin rod (the segment $(0,1)$) that had an initial temperature distribution $\sin(\pi x)$, i.e. it was warmer in the middle and coldest on the ends. The boundary conditions $u(0) = 0$ and $u(1) = 0$ can be thought of as attaching a thermocouple at each end that holds the end temperature at 0 degrees.

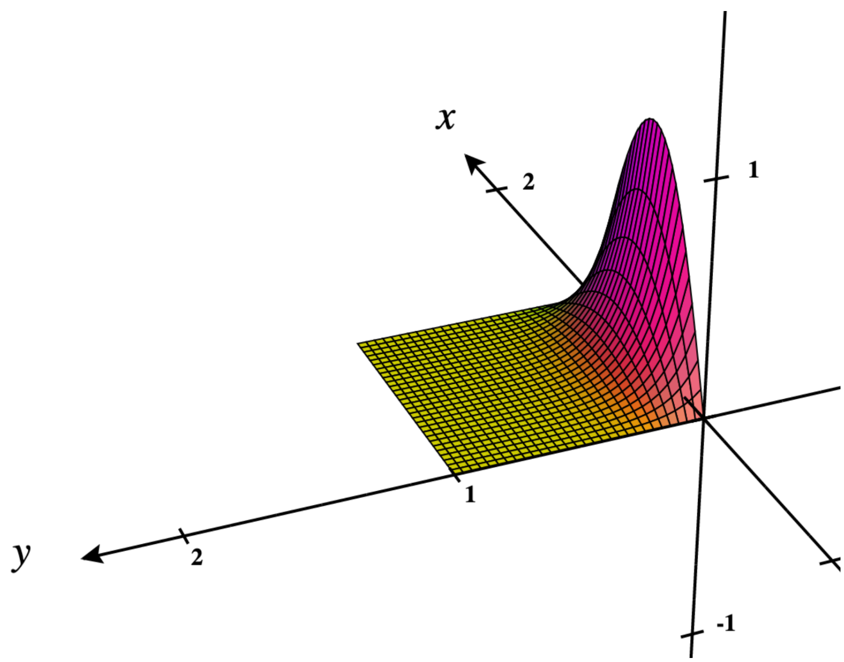
(a) Plot the function on CalcPlot3D by plotting

$$z = e^{-\pi^2 y}\sin(\pi x) = u(x,y),$$

where we just let the t variable be denoted by y to plot this function.

(b) Can you explain what happens as time t moves forward based on your intuition, plot, or by the equation we found in 4?

Solution 5. (a) Here is the plot.



(b) As t increases, the temperature starts to regularize in the rod. We can see that as t gets very large, we expect the temperature in the whole rod to be 0. This is fairly intuitive. If we old the ends of a rod at some temeprature, we do expect the temeprature to equalize throughout the rod given enough time.

Using the solution equation, we can just see that $e^{-\pi^2 t}$ gets smaller and smaller as t increases.