MATH 255, Homework 11

Problem 1 and 2 are related.

Problem 1. Consider the following linear system

$$x'(t) = x - y$$
$$y'(t) = -x - y.$$

(a) Rewrite this as a matrix equation

$$\mathbf{v}' = M\mathbf{v}.$$

Here the vector $\mathbf{v}$ denotes the xy-position of a particle at time t.

(b) Plot the vector field $\mathbf{v}'$.

(c) Describe what happens if your initial data is

i. $(x_0, y_0) = (0, 0)$,
ii. $(x_0, y_0) = (1, 1)$,
iii. $(x_0, y_0) = (-1, -1)$.

***Problem 2.** With the same linear system as in 1, do the following.

(a) Compute the eigenvalues of the matrix M.

(b) Compute the eigenvectors of the matrix M.

(c) Write the general solution for this system.

(d) Find the particular solution corresponding to the initial data $(x_0, y_0) = (1, 1)$.

Feel free to use Wolfram Alpha to do parts of this. However, you should be able to find eigenvalues on your own!

Problem 3. Solving the one dimensional Laplace equation is much like an ODE. However, the data given looks a bit different. So consider the following set up.

Consider the Laplace equation

$$\Delta u(x) = \frac{d^2u}{dx^2} = 0$$

on the interval $\Omega = (0, 1)$ with boundary conditions $u(0) = 0$ and $u(1) = 1$.

(a) This equation is separable. To find u, take two antiderivatives of

$$\frac{d^2u}{dx^2} = 0.$$

(b) To verify you did this correctly, take two derivatives of your function to see that you get 0.

(c) Your function should have two undetermined constants. Solve for these constants using the boundary conditions provided.

Problem 4. The methods for solving many PDEs are beyond the scope of this class, but we can still see what solutions behave like and a bit of how to find these. What we'll do below are a few steps of the method of *separation of variables* (not to be confused with separable ODE!)

Consider the heat equation in one dimension on the region $\Omega = (0, 1)$

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0$$

with boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$, and initial condition $u(x,0) = \sin(\pi x)$.

(a) Show that $f(x) = \sin(\pi x)$ is a solution to

$$f''(x) = -\pi^2 f(x)$$

with $f(0) = 0$ and $f(1) = 0$.

(b) Show that $g(t) = e^{-\pi^2 t}$ is a solution to

$$g'(t) = -\pi^2 g(t)$$

with $g(0) = 1$.

(c) Show that $u(x,t) = f(x)g(t)$ solves the heat equation with these boundary and initial conditions.

Problem 5. With our solution from 4, we can analyze the behavior of the system. The physical phenomenon that Problem 4 modelled was a thin rod (the segment $(0, 1)$) that had an initial temperature distribution $\sin(\pi x)$, i.e. it was warmer in the middle and coldest on the ends. The boundary conditions $u(0) = 0$ and $u(1) = 0$ can be thought of as attaching a thermocouple at each end that holds the end temperature at 0 degrees.

(a) Plot the function on CalcPlot3D by plotting

$$z = e^{-\pi^2 y} \sin(\pi x) = u(x,y),$$

where we just let the t variable be denoted by y to plot this function.

(b) Can you explain what happens as time t moves forward based on your intuition, plot, or by the equation we found in 4?