MATH 255, Homework 10: *Solutions*

Problem 1. Imagine a stream of water on the surface of a river that carries a particle along. The stream velocity is given by

$$\mathbf{v}(x, y) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

with v_1 and v_2 constant. We want to find the trajectory of the particle that begins at the point (x_0, y_0) at time $t = 0$.

(a) Let $\mathbf{x}(t)$ give the position of the particle at time t. Write the differential equation that satisfies the following statement:

At each time t, the particle's velocity is equal to the velocity of the stream at that point.

(b) Find the particular solution $\mathbf{x}(t)$ that satisfies our initial condition $\mathbf{x}(0) = (x_0, y_0)$.

(c) Now let $(v_1, v_2) = (5, 2)$ and $(x_0, y_0) = (1, 2)$. What is the particular solution?

The advantage to solving problems in this way (i.e., solving as we do in (b) and then using this to solve (c)) is that we have solved a problem that can be changed to suit another future problem. Leaving parameters in problems is highly advantageous.

Solution 1.

(a) The differential equation described here is to take the particle's velocity $\mathbf{x}'(t)$ (the derivative of the position $\mathbf{x}(t)$) and set it equal to the stream velocity $\mathbf{v}$. So we have

$$\mathbf{x}' = \mathbf{v}.$$

Specifically, we have

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

(b) We have two uncoupled equations

$$\begin{aligned} x' &= v_1, \\ y' &= v_2, \end{aligned}$$

which can both be integrated to find

$$\begin{aligned} x(t) &= v_1 t + c_1, \\ y(t) &= v_2 t + c_2. \end{aligned}$$

Noting that we want

$$\mathbf{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

we get

$$x(0) = x_0 = v_1(0) + c_1$$

which means $c_1 = x_0$. Similarly,

$$y(0) = y_0 = v_2(0) + c_2$$

which means $c_2 = y_0$. So our particular solution is

$$\mathbf{x}(t) = \begin{bmatrix} v_1 t + x_0 \\ v_2 t + y_0 \end{bmatrix}.$$

(c) We simply plug in these values and find

$$\mathbf{x}(t) = \begin{bmatrix} 5t + 1 \\ 2t + 2 \end{bmatrix}.$$

<u>Problems 2-4 are all related.</u>

Problem 2. Consider the nonlinear second order ODE

$$x'' - (1 - x^2)x' + x = 0.$$

This is known as the *Van der Pol oscillator*. It models a certain type of mass/spring system. In this case, I have set a parameter $\mu = 1$ for simplicity.

(a) Why is this equation nonlinear?

(b) Letting $y = x'$, rewrite this equation as a system of two first order ODE. That is, one equation for x' and the other for y'. *Note that x' is a velocity.*

(c) Plot the vector field given by this system. *Note that this vector field lives in <u>phase space</u>. That is, one axis is position and the other is velocity.*

(d) Using the vector field plot and the ODE itself, predict how this system evolves over time.

Solution 2.

(a) This equation is nonlinear since it cannot be written as

$$x'' + f(t)x' + g(t)x = h(t).$$

Specifically, there is the term

$$(1 - x^2)x'$$

that prevents this equation from being linear.

(b) We let $y = x'$ which means that we have $y' = x''$. If we substitute these into our ODE, we have

$$y' - (1 - x^2)y + x = 0$$

is first order in y. Thus, our system of ODE is

$$x' = y, \tag{1}$$
$$y' = (1 - x^2)y - x. \tag{2}$$

(c) The vector field we plot is given by the system we found. Specifically, plot

$$\mathbf{x}' = \begin{bmatrix} y \\ (1-x^2)y - x) \end{bmatrix}.$$

This plot looks like

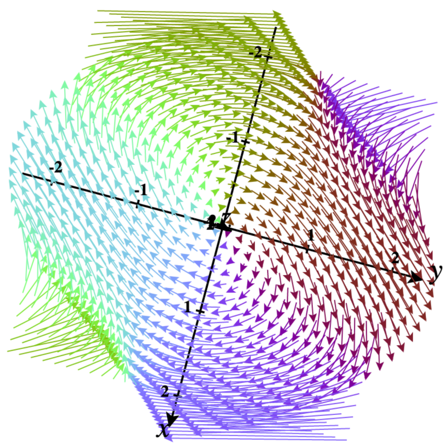

Figure 1: Vector field for the Van der Pol oscillator.

(d) The differential equation is similar to the Harmonic oscillator, but with an extra x' term. Based on this, and the vector field, it seems like there is due to be oscillatory behavior for this system. Whether or not trajectories grow or shrink in magnitude is not immediately obvious.

Problem 3. Now that we have rewritten the second order equation as system of first order ODE, we now want to solve this problem. To solve this, we need to linearize.

(a) Taking your planar system from Problem 2, linearize the system about the point $(0, 0)$.

(b) Taking same nonlinear system, linearize instead about the point $(1, 1)$.

(c) What are differences in the matrices you got for (a) and (b)?

Solution 3.

(a) From 2 we have,

$$x' = y,$$
$$y' = (1-x^2)y - x.$$

We must compute the partial derivatives

$$\frac{\partial x'}{\partial x} = 0, \qquad \frac{\partial x'}{\partial y} = 1,$$
$$\frac{\partial y'}{\partial x} = -2xy - 1, \qquad \frac{\partial y'}{\partial y} = 1 - x^2.$$

Evaluating these partials at the point $(0,0)$ gives us the linearization matrix

$$M_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

(b) Instead, at the point $(1,1)$, we evaluate these partials and get the linearization matrix

$$M_{(1,1)} = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}.$$

(c) The matrix $M_{(1,1)}$ has one more zero than the matrix $M_{(0,0)}$. This will affect the dynamics quite a bit.

Problem 4. Let's concentrate on solving the linear problem about each of the above points. Your linearizations around $(0,0)$ and $(1,1)$ takes the form

$$\mathbf{v}' = M\mathbf{v},$$

where M is a 2×2-matrix and

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) Find the eigenvalues of the matrix found by linearizing around $(0,0)$.

(b) Find the eigenvalues of the matrix found by linearizing around $(1,1)$.

(c) What is the general solution for the linear system about $(0,0)$?

(d) What is the general solution for the linear system about $(1,1)$?

(e) How do these different linearizations behave? Are they the same? If not, what is different?

Solution 4.

(a) The eigenvalues of the matrix $M_{(0,0)}$ are

$$\lambda_1 = \frac{1}{2}(1 + i\sqrt{3}), \qquad \lambda_2 = \frac{1}{2}(1 - i\sqrt{3}).$$

(b) The eigenvalues of the matrix $M_{(1,1)}$ are

$$\lambda_1 = i\sqrt{3}, \qquad \lambda_2 = -i\sqrt{3}.$$

(c) The general solution about $(0,0)$ is then

$$x(t) = Ae^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) + Be^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right).$$

(d) The general solution about $(1,1)$ is then

$$x(t) = A\sin(\sqrt{3}t) + B\cos(\sqrt{3}t).$$

Problem 5. One other common type of ordinary differential equation are the *exact* equations. They are very much related to potential fields that we have previously discussed. This form shows up more often in thermodynamics just due to the nature of the problems.

Let our differential equation be

$$x' = \frac{-x-1}{t+1}.$$

(a) Rewrite this equation in the form

$$P(x,t)dx + Q(x,t)dt = 0.$$

(b) Show this equation is exact by showing

$$\frac{\partial P}{\partial t} - \frac{\partial Q}{\partial x} = 0.$$

(c) Since this equation is exact, we can integrate $P(x,t)$ with respect to x determine the solution function up to some additional function of just t, $f(t)$.

(d) Repeat this, but integrate $Q(x,t)$ with respect to t to determine the solution function up to up some additional function of just x, $g(x)$.

(e) This you should now have determined an $F(x,t)$ that is correct up to some constant. Set this $F(x,t)$ (with the constant) equal to 0 and solve for x to find $x(t)$. This is your solution.

Solution 5.

(a) We begin with

$$x' = \frac{dx}{dt} = \frac{-x-1}{t+1}.$$

So we put it in the form as shown.

$$\begin{aligned}
\frac{dx}{dt} &= \frac{-x-1}{t+1} \\
dx &= \frac{-x-1}{t+1}dt \\
(t+1)dx &= (-x-1)dt \\
(t+1)dx + (-x-1)dt &= 0.
\end{aligned}$$

So, $P(x,t) = t+1$ and $Q(x,t) = x+1$.

(b) We compute

$$\frac{\partial P}{\partial t} = 1, \quad \text{and} \quad \frac{\partial Q}{\partial x} = 1.$$

Thus it follows that

$$\frac{\partial P}{\partial t} - \frac{\partial Q}{\partial x} = 1 - 1 = 0.$$

(c) We integrate

$$\int P(x,t)dx = \int t + 1dx = xt + x + f(t).$$

(d) Again, integrate

$$\int Q(x,t)dt = \int x + 1dt = xt + t + g(x).$$

(e) So, both the above functions must be equal. In fact, we have

$$F(x,t) = xt + x + f(t) = xt + t + g(x).$$

Now, we can see that

$$\begin{aligned} xt + x + f(t) &= xt + t + g(x) \\ x + f(t) &= t + g(x) \\ f(t) - g(x) &= t - x. \end{aligned}$$

So we can see that $f(t) = t$ and $g(x) = x$ from here. However, we really have only determined $F(x,t)$ up to some constant c, thus

$$F(x,t) = xt + x + t + c.$$

Now, we set $F(x,t) = 0$ to find our solution. So we have

$$\begin{aligned} F(x,t) &= 0 \\ xt + x + t + c &= 0 \\ xt + x &= -t - c \\ x(t+1) &= -t - c \\ x &= \frac{-t-c}{t+1}. \end{aligned}$$